The unreasonable effectiveness of Randomized Quasi-Monte Carlo in option pricing and risk analysis

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Option pricing problem can be formulated as

$$I[f] = \mathbb{E}[f(x)] = \int_{H^d} f(x)dx, \tag{1}$$

$\mathbb{E}[.]$ is the mathematical expectation.

f the payoff function, integrable in the d-dimensional unit hypercube $H^d = [0,1]^d$.

The standard MC estimator:

$$I_N[f] = \frac{1}{N}\sum_{i=1}^{N} f(x_i), \tag{2}$$

a.s. $I_N[f] \longrightarrow I[f]$. CLT provides confidence intervals:

$$I[f] \in [I_N[f] \pm c\frac{\hat{\sigma}_N}{\sqrt{N}}] \tag{3}$$

$\rightarrow$ In many financial applications, Quasi Monte Carlo (QMC) outperforms Monte Carlo

- showing faster empirical convergence rate: $I_N[f] \longrightarrow I[f]$ as rate $O(N^{-\alpha})$ with $0.5 < \alpha \leq 1$
- more stable convergence.

However, QMC lacks a practical error estimate.

$\rightarrow$ Randomized QMC (RQMC) method by randomizing the LDS points $\{x_i\}$ combines the best of two methods:

- it allows to compute confidence intervals around the estimated value as in MC;
- It may further improve the convergence rate

- Randomized QMC;
- Different Sobol' sequence generators;
- Hyperbolic local volatility model;
- Standard, Brownian Bridge, PCA discretization schemes;
- Results MC, QMC, RQMC pricing and Greeks computation of Asian options;
- Global Sensitivity Analysis, Effective dimensions.

Generate a set of n LDS points $\{Q_i\}$,

Generate a set of K randomised replication of $\{Q_i\}$: $\{V_i\} = V_i^k, k = 1, ..., K$.

Define $\hat{\mu}_n^k$ - the $k-th$ RQMC estimator for (1):

$$\hat{\mu}_n^k = \frac{1}{n} \sum_{i=1}^{n} f(V_i^k), \tag{4}$$

$\hat{\mu}_n^k$ are i.i.d. random variables

The RQMC sample mean

$$\bar{\mu}_n = \frac{1}{K} \sum_{k=1}^{K} \hat{\mu}_n^k. \tag{5}$$

Table: MC and RQMC sample standard deviations σ, RMSE errors ε and confidence intervals. The total number of function evaluations $N = nK$.

$\sigma_{MC} = \sqrt{\frac{1}{(N-1)} \sum_{i=1}^{N} (f(X_i) - I_N[f])^2}$	$\sigma_{RQMC} = \sqrt{\frac{1}{(K-1)} \sum_{k=1}^{K} (\hat{\mu}_n^k - \bar{\mu}_n)^2}$
$\varepsilon_{MC} = \frac{\sigma_{MC}}{\sqrt{N}}$	$\varepsilon_{RQMC} = \frac{\sigma_{RQMC}}{\sqrt{K}}$
$I_N[f] \pm z_{\delta/2} \varepsilon_{MC}$	$\bar{\mu}_n \pm z_{\delta/2} \varepsilon_{RQMC}$

K is large enough - $\bar{\mu}_n \sim N(I[f], \sigma_{RQMC})$.

z_δ is the $1-\delta$ quantile of the standard normal distribution: $F(z_\delta) = 1 - \delta$.

"All Sobol' sequence generators are equal but some are more equal than others"

The efficiency of Sobol' LDS generators depend on direction numbers.

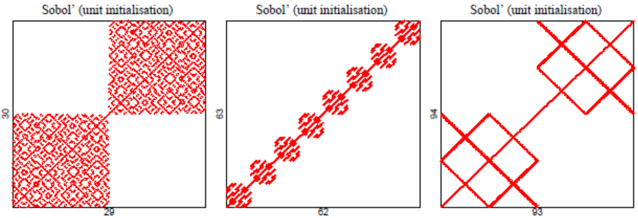


Figure: Sobol' sequences with badly initialized direction numbers.[1]

Joe&Kuo's generator - 'optimized' 2D projections (maximum dim. d =21201)[2].

BRODA's SobolSeq - additional uniformity properties[3]:
1) Property A for all dimensions (maximum dim. d =131072)
2) Property A' for adjacent dimensions.

[1] P. Jackel, Monte Carlo Methods in Finance, John Wiley&Sons, 2002
[2] S.Joe, F.Y.Kuo. SIAM J. Scientific Comp., 30, 2635-2654, 2008
[3] I. Sobol', D. Asotsky, A. Kreinin, S. Kucherenko. 2011, Wilmott Journal, Nov, 64-79

Consider $\mathbf{z} = (z_1, \dots, z_d) \sim \mathcal{N}(0, I)$, and define $\bar{z}_d = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} z_i$.

$z_i = F^{-1}(x_i)\,, x_i{}'\mathrm{s}-$ are Sobol' points: $\{x_i\}_{i=1}^{N} \subset (0,1)^d$

Example: A terminal asset value $S(T)$ in the case of d time steps

$$S(T) = S_0 \exp\left[\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{\Delta t}\left(z_1 + z_2 + \dots + z_d\right)\right] = \tag{6}$$

$$= S_0 \exp\left[\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}\bar{z}_d\right] \tag{7}$$

Here the Wiener path is sampled using the Standard (incremental) discretization scheme.

Consider variance $V\left(\bar{z}_d\right)$, assuming that $E\left(\bar{z}_d\right) = 0$:

$$V\left(\bar{z}_d\right) = \frac{1}{d}\sum_{i=1}^{d}\sum_{j=1}^{d}\rho_{ij} = \frac{1}{d}\left[\sum_{i=1}^{d} 1 + \sum_{i=1}^{d}\sum_{j:j\neq i}^{d}\rho_{ij}\right] = 1 + \bar{\rho}_d. \quad (8)$$

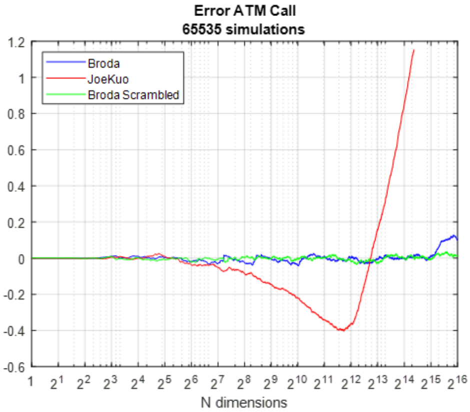
$$\bar{\rho}_d = \frac{1}{d}\sum_{i=1}^{d}\sum_{j:j\neq i}^{d}\rho_{ij} \text{ is an average correlation, } \rho_{ij} = E\left[z_i z_j\right]. \quad (9)$$

Theoretically $\rho_{ij} = 0, i \neq j$, hence $V\left(\bar{z}_d\right) = 1$.

Numerically $\bar{\rho}_d \neq 0$ due to the presence of spurious correlations between different dimensions of actual LDS sequences. We call $\bar{\rho}_d$ - a "spurious variance component"

European call: $S_0 = 100$, K= 100, r = 0.0, $\sigma = 0.2$, T = 1y. $C_{BS} = 7.966$.

$$C = e^{-rT} \int_{H^d} \max[0, (S_0 \exp[(r - \frac{\sigma^2}{2})T + \sigma\sqrt{\frac{T}{d}}\sum_{j=1}^{d} F^{-1}(u_j)] - K)]du_1...du_d$$



BRODA - SobolSeq generator; BRODA Scrambled - Owen's scrambling with additional permutations; Joe-Kuo - Direction numbers of Joe&Kuo

Joe-Kuo's generator has unacceptably high spurious variance components.

A set d-dimensional Sobol' points $\{Q_i\}$ in base $b = 2$

$$Q_i^j = \sum_{p=1}^{m} q_{i,p}^j 2^{-p}, \tag{10}$$

Generate r.n. $U \sim U[0,1]^d$, $U^j = \sum_{p=1}^{m} u_p^j 2^{-p}$

Randomised version with Digital shift (DS):

$$V_i^j = \sum_{p=1}^{m} v_{i,p}^j 2^{-p} : \tag{11}$$

$$v_{i,p}^j = (q_{i,p}^j \oplus u_p^j) \tag{12}$$

$\oplus$ - binary addition modulo 2 (a bitwise XOR operator):
$0 \oplus 0 = 0$; $1 \oplus 1 = 0$; $0 \oplus 1 = 1$; $1 \oplus 0 = 1$

Method	Pros	Cons
Owen's Scrambling[4]	Higher rate: $\varepsilon_{RQMC} \sim O(1/n^{(3/2-\alpha)})$ $\sqrt{n}$ faster than QMC: $O\left(1/n^{(1-\alpha)}\right)$;	High CPU/memory; Permutation tree size $\Pi \sim d(b^M - 1)/(b-1)$
Digital Shift	Simple to implement No extra memory	No increased rate of convegence

Owen's scrambling is the most efficient sampling method. Accuracy is improved by random error cancellations.

Note: BRODA's modification of Owen's scrambling[5] reduces memory and CPU demands.

[4] A. Owen, Ann. Stat., 25(4):1541, 1997

[5] E. Atanassov, S. Kucherenko, BRODA Ltd., UK, 2021

Comparison of Sobol', Digital Shift and Owen scrambling in 2D

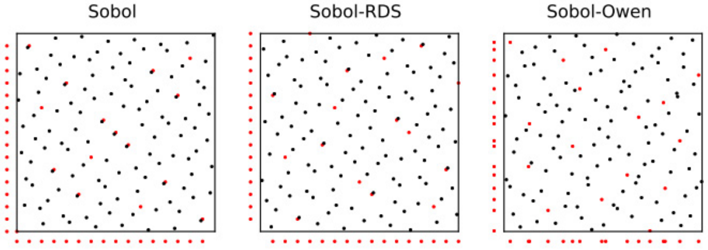


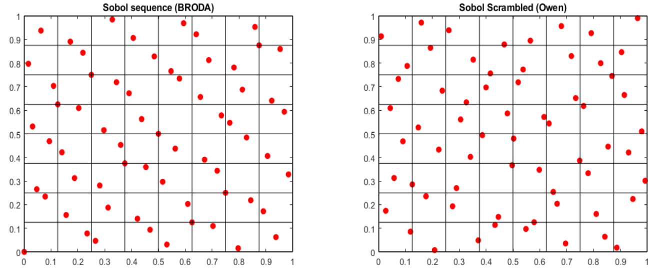
Fig. First 128 points of 2D Sobol sequence[6]. First 16 points colored red.

Left: Sobol points: stratified 1D spacing. *Aligned on 2D diagonals.*

Middle: Digit scrambling: offsets 1D. *2D structure unchanged.*

Right: Owen scrambling: *jitters 1D/2D, preserves stratification*.

[6] B. Burley, Practical Hash-based Owen Scrambling. J. Comp. Graphics Tech., 9 (4) 2020



(a) SobolSeq (b) SobolOwen

Video: Standard and Scrambled Sobol Sequences

Comparison of Random numbers and Sobol Sequences – YouTube, BRODA

Difference between Standard and Scrambled Sobol Sequences – YouTube, BRODA

Asset prices follow:

$$dS_t = rS_t dt + \tilde{\sigma}(S_t)dW_t,\ S_0 = 1, \tag{13}$$

with the risk free interest rate r and time-homogeneous hyperbolic local volatility (HLV) model[7]:

$$\tilde{\sigma}(S) = \nu \Big\{ \frac{(1-\beta+\beta^2)}{\beta} S_t + \frac{(\beta-1)}{\beta} \Big(\sqrt{S_t^2 + \beta^2 (1-S_t)^2} - \beta \Big) \Big\}, \tag{14}$$

Here $\nu > 0$ - the level of volatility; $\beta \in (0,1]$ - the skew parameter.

Euler time discretization of the SDE

$$dY(t) = [r - \frac{1}{2}\sigma^2(Y(t))]dt + \sigma(Y(t))dW_t,\ Y(0) = \log(S(0)), \tag{15}$$

with $Y(t) = \ln(S(t))$ and $\sigma(Y) = \frac{\tilde{\sigma}(e^Y)}{e^Y}$.

$$Y^n(t_{i+1}) = Y^n(t_i) + [r - \frac{1}{2}\sigma^2(Y^n(t_i))](t_{i+1} - t_i) + \sigma(Y^n(t_i))(W(t_{i+1}) - W(t_i)) \tag{16}$$

with $Y^n(0) = \log(S(0))$, $\Delta t = \frac{T}{d}, t_i = i\Delta t,\ i = 0, .., d$.

[7] Jaeckel, P. 2008. Hyperbolic local volatility. http://www.jaeckel.org/

Brownian path $W = (W_1, W_2, \dots, W_\mathrm{d})^\mathrm{T}$ is normally distributed: $E(W) = 0$ and cov. matrix $\Sigma = [\min(t_i, t_j)]_{i,j=1}^{d}$
The value of options can be written as

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}((f(S,K)) =& \mathbb{E}^{\mathbb{Q}}((f(W,K)) && (17)\\
=& \int_{\mathbb{R}^\mathrm{d}} P(W) \frac{\exp\left(-\frac{1}{2} W^\mathrm{T} \Sigma^{-1} W\right)}{\sqrt{(2\pi)^\mathrm{d} \det(\Sigma)}} \mathrm{d}W && (18)\\
=& \int_{\mathbb{R}^\mathrm{d}} P(Lz) \frac{\exp\left(-\frac{1}{2} z^\mathrm{T} z\right)}{\sqrt{(2\pi)^\mathrm{d}}} \mathrm{d}z && (19)\\
=& \int_{[0,1]^\mathrm{d}} P\left[\ \mathrm{L}\Phi^{-1}(x)\right] \mathrm{d}x, && (20)\\
=& \int_{[0,1]^\mathrm{d}} f(x) \mathrm{d}x && (21)
\end{aligned}
$$

where we use the change of variable $W = Lz$ in the second equality and for the last equality, the mapping $z = \Phi^{-1}(x)$ with $\Phi^{-1}(x)$ the inverse of the standard normal cumulative distribution function (applied elementwise). Here $\Sigma = LL^\mathrm{T}$.

We will consider 3 different ways of $\Sigma = LL^{\mathrm{T}}$ decomposition:

- Cholesky factorization or standard discretization:

$$W(t_i) = W(t_{i-1}) + \sqrt{\Delta t} Z_i \;\; 1 \leq i \leq d, \tag{22}$$

Z_i independent standard normal variates. Cholesky factorisation of $\Sigma = LL^{\mathrm{T}}$.

- Brownian bridge (BB) algorithm:
  1. First we generate the variable at the terminal point

$$W(T) = \sqrt{T} Z_1 \tag{23}$$

  2. Then we fill other points using already found values of $W(t_i)$

$$W(t_i) = (1-\gamma)W(t_l) + \gamma W(t_m) + \sqrt{\gamma(1-\gamma)(m-l)\Delta t} Z_i, \tag{24}$$

  where $\gamma = \frac{i-l}{m-l}$ with $l \leq i \leq m$. It can be seen from equation (24) that the variance of the stochastic part of the BB formula decreases rapidly at the successive levels of refinement and the first few points contain most of the variance.

- PCA

$$L = U\Lambda^{1/2} = (U_1\,|U_2|\ldots)\begin{pmatrix} \lambda_1^{1/2} & & \\ & \lambda_2^{1/2} & \\ & & \ldots \end{pmatrix}. \tag{25}$$

where:

- U is an orthogonal matrix whose columns are the corresponding unit eigenvectors (i.e., $U^\top U = I_d$),
- $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_d)$ the diagonal matrix of eigenvalues of Σ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0$.

PCA selects the transformation matrix L such that the first principal component U_1 captures the maximum variance of W; the second component U_2 captures the maximum remaining variance, conditional on U_1; and subsequent components $(U_3, \ldots, U_d)$ are determined iteratively.

Geometric average Asian call option payoff:

$$P_A = \max(\bar{S} - K, 0), \tag{26}$$

with $\bar{S} = \left(\prod_{i=1}^{n} S_i\right)^{\frac{1}{d}}$, where $S_i = S(t_i)$, $t_i = i\frac{T}{d}$, $1 \leq i \leq d$.

Price:

$$AC(T,K) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[P_A] \approx AC_N(T,K) = e^{-rT}\left[\frac{1}{N}\sum_{l=1}^{N}\max(\bar{S}^{(l)} - K, 0)\right], \tag{27}$$

where $\bar{S}^{(l)}$ is $\bar{S}$ at the simulated price paths l.

Sensitivities Δ and Γ defined and approximated as :

$$\Delta := \frac{\partial AC(T,K)}{\partial S(0)} \approx \frac{AC_N(T,K,S(0)+\epsilon_s) - AC_N(T,K,S(0)-\epsilon_s)}{2\epsilon_s} \tag{28}$$

$$\Gamma := \frac{\partial^2 AC(T,K)}{\partial S_0^2} \approx \frac{AC_N(T,K,S(0)+\epsilon_s) + AC_N(T,K,S(0)-\epsilon_s) - 2AC_N(T,K,S(0))}{\epsilon_s^2} \tag{29}$$

Recall:

Table: MC and RQMC sample standard deviations σ, RMSE errors ε and confidence intervals. The total number of function evaluations $N = nK$.

$\sigma_{MC} = \sqrt{\frac{1}{(N-1)} \sum_{i=1}^{N} (f(x_i) - \hat{\mu}_N)^2}$	$\sigma_{RQMC} = \sqrt{\frac{1}{(K-1)} \sum_{k=1}^{K} (\hat{\mu}_n^k - \bar{\mu}_n)^2}$
$\varepsilon_{MC} = \frac{\sigma_{MC}}{\sqrt{N}}$	$\varepsilon_{RQMC} = \frac{\sigma_{RQMC}}{\sqrt{K}}$

Table: ε_{MC}, $\varepsilon_{RQMC-BB}$ and $\varepsilon_{RQMC-PCA}$ of price estimations.

	ITM	ATM	OTM
ε_{MC}	3.26 10^{-2}	2.2 10^{-2}	1.24 10^{-3}
$\varepsilon_{RQMC-BB}$	3.67 10^{-4}	6.09 10^{-4}	4.04 10^{-4}
$\varepsilon_{RQMC-PCA}$	3.2 10^{-4}	3.11 10^{-3}	3.05 10^{-4}
$\varepsilon_{MC}/\varepsilon_{RQMC-BB}$	89	36	3
$\varepsilon_{MC}/\varepsilon_{RQMC-PCA}$	102	71	4

Numerical results: Confidence intervals for prices and Greeks

Table: ε_{MC}, $\varepsilon_{RQMC-BB}$ and $\varepsilon_{RQMC-PCA}$ of Deltas estimations.

	ITM	ATM	OTM
ε_{MC}	3.29 10^{-4}	2.43 10^{-3}	4.09 10^{-4}
$\varepsilon_{RQMC-BB}$	3.23 10^{-5}	5.98 10^{-4}	1.41 10^{-4}
$\varepsilon_{RQMC-PCA}$	3.25 10^{-5}	4.91 10^{-4}	4.05 10^{-4}
$\varepsilon_{MC}/\varepsilon_{RQMC-BB}$	10	4	3
$\varepsilon_{MC}/\varepsilon_{RQMC-PCA}$	10	49	10

Table: ε_{MC}, $\varepsilon_{RQMC-BB}$ and $\varepsilon_{RQMC-PCA}$ of Gammas estimations.

	ITM	ATM	OTM
ε_{MC}	4.41 10^{-5}	9.08 10^{-4}	2.24 10^{-4}
$\varepsilon_{RQMC-BB}$	3.72 10^{-5}	6.79 10^{-4}	1.82 10^{-4}
$\varepsilon_{RQMC-PCA}$	2.20 10^{-5}	1.33 10^{-4}	4.62 10^{-5}
$\varepsilon_{MC}/\varepsilon_{RQMC-BB}$	1	1	1
$\varepsilon_{MC}/\varepsilon_{RQMC-PCA}$	2	7	5

Power law integration error approximation:

$$\varepsilon_n \sim \frac{C}{n^{\alpha}} \ . \tag{30}$$

Use the RMSE below to approximate the rate of convergence

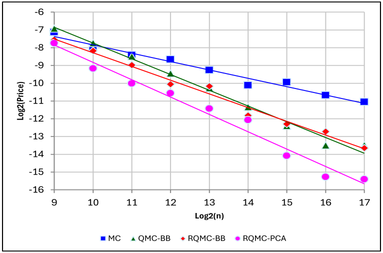
$$\varepsilon_n = \sqrt{\frac{1}{K}\sum_{k=1}^{K}\left(V - V_n^{(k)}\right)^2}, \tag{31}$$

V is the reference Price or Greek values.

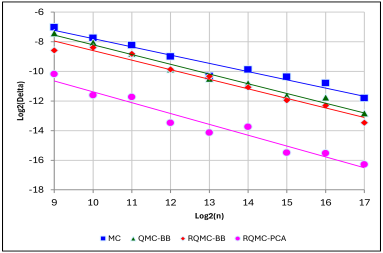
	ITM	**ATM**	**OTM**
RQMC BB (Price)	1.0	0.8	0.77
RQMC PCA (Price)	1.0	0.91	0.97
RQMC BB (Delta)	0.7	0.64	0.58
RQMC PCA (Delta)	0.88	0.73	0.87
RQMC BB (Gamma)	0.58	0.6	0.55
RQMC PCA (Gamma)	0.54	0.86	0.71

Table: Extracted α in RQMC BB and RQMC PCA

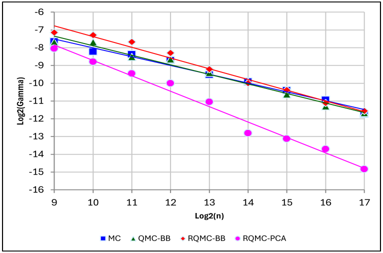
Performance analysis for ATM price (a), delta (b) and gamma (c)



(c)



(d)



(e)

Let's consider a square integrable $f(x)$ in $\mathcal{H}^d = [0,1]^d$. f has an unique ANOVA decomposition:

$$f(x) = f_0 + \sum_{i=1}^{n} f_i\left(x_i\right) + \sum_{i}\sum_{j>i} f_{ij}\left(x_i, x_j\right) + \ldots + f_{1,2,\ldots d}\left(x_1, x_2, \ldots, x_d\right) \quad (32)$$

if

$$f_0 = \int_{[0,1]^d} f(x)\, dx, \quad (33)$$

and for all $k = 1, 2, \ldots, s$

$$\int_0^1 f_{i_1,\ldots,i_s}(x_{i_1}, \ldots.., x_{i_s}) dx_{i_k} = 0. \quad (34)$$

Each ANOVA term $f_{i_1,\ldots,i_s}(x_{i_1}, \ldots.., x_{i_s})$ is a function of a unique subset of variables from x and the terms are orthogonal.

Variance decomposition:

$$\sigma^2 = \sum_{i=1}^{d} \sigma_i^2 + \sum_{i=1}^{d} \sum_{i<j}^{d} \sigma_{ij}^2 + ... + \sigma_{12...d}^2. \tag{35}$$

Here σ^2 is the total variance,

$$\sigma_{i_1,...,i_s}^2 = \int_0^1 f_{i_1,...,i_s}^2 (x_{i_1}, ..., x_{i_s}) dx_{i_1} ... dx_{i_s} \tag{36}$$

are called partial variances.

Sobol' Sensitivity Indices (SI):

$$1 = \sum_{i=1}^{n} S_i + \sum_{i<j} S_{ij} + \sum_{i<j<l} S_{ijl} + \ldots + S_{1,2,...,d} \tag{37}$$

where $S_{i_1,...,i_s} := \frac{\sigma_{i_1,...,i_s}^2}{\sigma^2}$.

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[f(S,K)] &= \mathbb{E}^{\mathbb{Q}}[f(W,K)] && (38)\\
&= \int_{\mathbb{R}^d} P(W)\,\frac{\exp\left(-\frac{1}{2}W^{\mathrm{T}}\Sigma^{-1}W\right)}{\sqrt{(2\pi)^d\det(\Sigma)}}\,\mathrm{d}W && (39)\\
&= \int_{\mathbb{R}^d} P(Lz)\,\frac{\exp\left(-\frac{1}{2}z^{\mathrm{T}}z\right)}{\sqrt{(2\pi)^d}}\,\mathrm{d}z && (40)\\
&= \int_{[0,1]^d} P\left[L\,\Phi^{-1}(x)\right]\,\mathrm{d}x && (41)\\
&= \int_{[0,1]^d} f(x)\,\mathrm{d}x. && (42)
\end{aligned}
$$

We transformed the option price into an expectation over independent uniforms. Which dimensions of x impact the price most?

The key is to analyze the integrand $f(x)$:

1. By computing its Sobol' indices, we identify the most influential drivers.
2. By calculating its mean effective dimension, we gain insight into the overall complexity of the problem.

Definition: The mean dimension of f is

$$\hat{d} = \sum_{u \subseteq 1:d} |u| S_u. \tag{43}$$

with $|\mathrm{u}|$ the cardinality of a set of variables u

Theorem (Owen 2011): $\hat{d} = \sum_{i=1}^{d} S_i^{\mathsf{tot}}$

with the total Sobol' index for input variable x_i defined as

$$S_i^{\mathsf{tot}} := \sum_{\substack{u \subseteq \{1,\dots,d\} \\ i \in u}} S_u := \frac{D_y^{\mathsf{tot}}}{\sigma^2} \tag{44}$$

It satisfies $1 \leqslant \hat{d} \leqslant d$. Various examples show that QMC outperforms MC integration if the integrand $f(x)$ has $\hat{d} < 3$

By denoting $x = (y, z)$, we have

Theorem (Sobol 2001): Subset's total variance D_y^{tot} is equal to

$$D_y^{\text{tot}} = \frac{1}{2} \int [f(y, z) - f(y', z)]^2 dx dy' \tag{45}$$

Let $\xi_j = (\eta_j, \zeta_j), \quad \xi'_j = (\eta'_j, \zeta'_j)$ independent random points uniformly ditributed in H^d with $j = 1, ..., N$. Then

$$\frac{1}{2N} \sum_{i=1}^{N} \left[f(\xi_j) - f(\eta'_j, \zeta_j) \right]^2 \xrightarrow{P} D_y^{\text{tot}}. \tag{46}$$

Table: Mean effective dimension $\hat{d}$ for Asian option price and Greeks, ITM

	BB	PCA	Standard
Price	**1.10**	**1.0**	**1.0**
Delta	**1.04**	**1.0**	1.22
Gamma	6.0	**1.80**	26.6

TABLE: Mean effective dimension $\hat{d}$ for Asian option price and Greeks, ATM

	BB	PCA	Standard
Price	**1.11**	**1.01**	**1.3**
Delta	**1.37**	**1.05**	3.22
Gamma	4.79	**1.80**	29.3

TABLE: Mean effective dimension $\hat{d}$ for Asian option price and Greeks, OTM

	BB	PCA	Standard
Price	**2.21**	**1.12**	**5.8**
Delta	**2.10**	**1.1**	4.28
Gamma	4.10	**1.62**	26.43

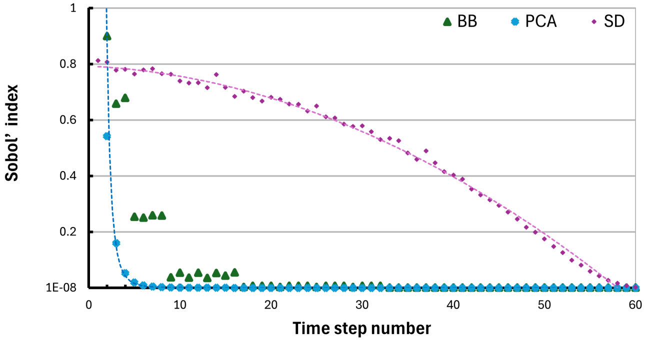


Figure: Total Sobol SI w.r.t variable x_i for Standard (SD), BB and PCA schemes

The initial coordinates of Sobol' sequences are much better distributed than the later high dimensional coordinates. The Brownian bridge and PCA discretizations use low well distributed coordinates from each d-dimensional LDS vector point to determine most of the structure of a path and reserves the later coordinates to fill in fine details.

- RQMC with Owen's scrambling outperforms MC and QMC, enabling reliable confidence intervals and faster convergence.
- Effective dimension reduction (PCA, Brownian Bridge) is the key driver of efficiency, yielding substantial accuracy gains in both pricing and Greeks.
- PCA-based RQMC consistently delivers the lowest RMSE, outperforming all other methods and making difficult sensitivities (especially Gamma) feasible and reliable.
- Effective dimension, not nominal dimension, determines success for high-dimensional, path-dependent financial simulations.

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Kuchenrenko, S and Hok, J. The importance of being scrambled: supercharged Quasi Monte Carlo. Journal of Risk, 26(1):1-20,2023.

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