

# The Importance of being Global. Application of Global Sensitivity Analysis in Monte Carlo Option Pricing

**Sergei Kucherenko, Nilay Shah**

Imperial College London, SW7 2AZ, UK,

e-mail: s.kucherenko@ic.ac.uk

## Abstract

Monte Carlo and Quasi Monte Carlo methods for pricing European and Asian call options are compared. Two different discretization schemes, namely the standard discretization and the Brownian Bridge construction are considered. Results show superior performance of the QMC approach based on Sobol' sequences with the Brownian Bridge discretization regardless of dimensionality. Global sensitivity

analysis is used for comparison and explanation the differences in performances of both schemes.

## Keywords

Monte Carlo, Quasi Monte Carlo, Global Sensitivity Analysis, Sensitivity Indices, Sobol' sequences, Brownian bridge

## 1 Introduction

Monte Carlo (MC) simulation is a common tool in valuation of complex financial instruments. It is popular because of the lack of analytical solutions for most financial models. The convergence rate of MC methods is  $O(1/\sqrt{N})$ , where  $N$  is the number of sampled points. It does not depend on the number of variables  $n$  although it is rather slow. A higher rate of convergence can be obtained by using deterministic uniformly distributed sequences known as low-discrepancy sequences (LDS) instead of pseudo-random numbers. Methods based on the usage of LDS are known as Quasi Monte Carlo (QMC). Asymptotically, QMC can provide a rate of convergence  $\sim O(1/N)$ .

The solution of many financial problems can be reduced to evaluation of the Wiener path integrals. To achieve a fast convergence for path integrals an important role is played by path discretization algorithms. Two widely used algorithms are known as the standard discretization algorithm and the Brownian bridge construction. Both algorithms were considered by Sobol' (1962, 1973). The Brownian bridge construction was analyzed by Moskowitz et. al. in 1996 within the framework of the QMC approach. Both algorithms have the same variance, hence their MC convergence rates are the same. However, the corresponding QMC algorithms have different efficiencies with the Brownian bridge having much higher convergence rate. It was suggested that the improvement in convergence is due to reduction of the effective dimension of the problem. The notion of the "effective dimension" was introduced in Caflisch et. al. (1997).

However, this measure of the quality of QMC methods is rather complex and it depends on an arbitrary confidence level. The approach presented in this paper is more general and practical. It is based on global sensitivity analysis (SA).

Global SA offers a comprehensive approach to model analysis. It enables the identification of key parameters whose uncertainty most affects the output. It then can be used to rank variables, fix unessential variables or decrease problem dimensionality. Global SA is a relatively new method but its efficiency and importance has been already recognized in many areas. However, it has hardly been applied in finance ( for references see Complongo et. al. 2006, Sobol' et. al. 2005).

Unlike local SA, global SA methods evaluate the effect of a factor while all other factors are varied as well and thus they account for interactions between variables and do not depend on the stipulation of a "nominal" point. The method of global sensitivity indices developed by Sobol' (1993, 2001) is based on ANOVA type of a high dimensional model representation. It is superior to other SA methods, such as those based on correlation or regression coefficients because it is model-independent; it allows the estimation of not only the individual contribution of each input parameter to the output variance but it also captures the interaction effects. It also allows the calculation of the total sensitivity indices, which measure the total contribution of a single input factor, and thus avoids the "curse of dimensionality".

Global SA can be used to access the efficiency of numerical schemes. In Sobol' et. al. (2005) global SA was applied for explaining the effectiveness

of the Brownian Bridge construction using a test problem. In Kucherenko et. al (2007) this technique was applied for explaining the effectiveness of high dimensional integration based on QMC methods.

The rest of this paper is organized as follows. The next section presents a brief review of the MC and QMC methods and LDS. Section 3 describes two schemes for discretization of the Wiener process and the MC approach for option pricing. Section 4 introduces global SA. In Section 5 the results of our experiments, the pricing of European and geometric average Asian options are presented. Both options have closed-form solutions, which makes them ideal for comparison purposes. We compare traditional MC and QMC methods based on standard and the Brownian bridge discretization schemes. The superior performance of the QMC approach is explained by comparing global sensitivity indexes for both discretization schemes. Finally, conclusions are presented in section 6.

## 2 MC and QMC algorithms

Consider the Wiener path integral:

$$I = \int_C F[x(t)] d_W x, \quad (2.1)$$

where  $C$  is the space of all functions  $x(t)$  continuous in the interval  $0 \leq t \leq T$  with a boundary condition  $x(0) = x_0$ . The integral (2.1) can be seen as an expectation with respect to the Wiener measure on  $C$ , so that  $I = E(F[W(t)])$ . Here  $W(t)$  is a random Wiener processes ( also known as a Brownian motion ). The MC approach in this case would consist of constructing many random paths  $W(t)$ , evaluating  $F[W(t)]$  and averaging the results. In addition, discretization algorithms for constructing  $W(t)$  are required. They are considered in the following section. Assuming that path  $W(t)$  is approximated by  $n$ -dimensional discretization scheme, (2.1) can be reduced to the multidimensional integral

$$I[f] = \int_{H^n} f(x) dx. \quad (2.2)$$

Here function  $f(x)$  is integrable in the  $n$ -dimensional unit hypercube  $H^n$ .

The MC quadrature formula is based on the probabilistic interpretation of an integral. For a random variable that is uniformly distributed in  $H^n$

$$I[f] = E[f(x)],$$

where  $E[f(x)]$  is the mathematical expectation. An approximation to this expectation is

$$I_N[f] = \frac{1}{N} \sum_{i=1}^N f(x_i), \quad (2.3)$$

where  $\{x_i\}$  is a sequence of random points in  $H^n$  of length  $N$ . The approximation  $I_N[f]$  converges to  $I[f]$  with probability 1.

Consider an integration error defined as

$$\varepsilon = |I[f] - I_N[f]|.$$

It follows from the Central Limit Theorem that the expectation of  $\varepsilon^2$  is

$$E(\varepsilon^2) = \frac{\sigma^2(f)}{N},$$

where  $\sigma^2(f)$  is the variance given by

$$\sigma^2(f) = \int_{H^n} f^2(x) dx - \left( \int_{H^n} f(x) dx \right)^2.$$

Then the expression for the root mean square error of the MC method is

$$\varepsilon_N = (E(\varepsilon^2))^{1/2} = \frac{\sigma(f)}{N^{1/2}}. \quad (2.4)$$

The convergence rate of MC does not depend on the number of variables  $n$  although it is rather low.

The efficiency of MC methods is determined by the properties of random numbers. It is known that random number sampling is prone to clustering: for any sampling there are always empty areas as well as regions in which random points are wasted due to clustering. As new points are added randomly, they do not necessarily fill the gaps between already sampled points.

LDS are specifically designed to place sample points as uniformly as possible. Unlike random numbers, successive LDS points “know” about the position of previously sampled points and “feel” the gaps between them. LDS are also known as quasi random numbers. The QMC algorithm for the evaluation of the integral (2.2) has a form similar to (2.3)

$$I_N = \frac{1}{N} \sum_{i=1}^N f(q_i). \quad (2.5)$$

Here  $\{q_i\}$  is a set of LDS points uniformly distributed in a unit hypercube  $H^n$ ,  $q_i = (q_i^1, \dots, q_i^n)$ .

There are a few well-known and commonly used LDS. Different principles were used for their construction by Holton, Faure, Sobol', Niederreiter and others. Many practical studies have proven that the Sobol' LDS is in many aspects superior to other LDS ( Jaeckel 2002, Wilmott 2005). For this reason it was used in this work.

Sobol' LDS were constructed by following the three main requirements (Sobol',1992 ):

1. Best uniformity of distribution as  $N \rightarrow \infty$ .
2. Good distribution for fairly small initial sets.
3. A very fast computational algorithm.

Points generated by the Sobol' LDS produce a very uniform filling of the space even for a rather small number of points  $N$ , which is a very important case in practice.

For the best known sequences  $\{q_i\}$  the estimate for the rate of convergence  $I_N \rightarrow I$  is known to be  $O(\ln^n N)/N$ . This rate of convergence is much faster than that for the MC method (4), although it depends on the dimensionality  $n$ . Consequently, the smaller  $n$ , the better this estimate. In practice at  $n > 1$  the rate of convergence  $O(\ln^n N)/N$  is not observed. It appears to be approximately  $N^{-\alpha}$ ,  $0 < \alpha \leq 1$ . For financial problems  $0.5 < \alpha \leq 1$ . Hence, the QMC method always outperforms MC in terms of convergence. As shown in the following sections,  $\alpha$  can be dramatically increased by using so-called effective dimension reduction techniques.

## 3 Option pricing

### 3.1 Discretization of the Wiener process

Consider the problem of pricing an option on a single asset whose value at time  $t$  is denoted by  $S(t)$ . We assume the asset follows a geometric Brownian motion process:

$$dS = \mu S dt + \sigma S dW. \quad (3.1)$$

Here  $\mu$  is the drift rate,  $\sigma$  is the volatility,  $t$  is time,  $W$  is the standard Brownian motion:

$$dW = Z(dt)^{1/2}, \quad (3.2)$$

where  $Z$  is a normally distributed random variable  $N(0,1)$  with mean 0 and variance 1. Using Ito's lemma and the risk neutral measure assumption equations (3.1), (3.2) with constant coefficients  $r$  and  $\sigma$  can be solved analytically

$$S(t) = S_0 \exp \left[ \left( r - \frac{1}{2} \sigma^2 \right) t + \sigma W(t) \right].$$

Here  $W(t)$  is a Wiener path up to time  $t$  starting at  $W(t=0) = 0$ . One time step  $\Delta t$  can be simulated using equation

$$S(t + \Delta t) = S(t) \exp \left[ \left( r - \frac{1}{2} \sigma^2 \right) \Delta t + \sigma (W(t + \Delta t) - W(t)) \right]. \quad (3.3)$$

In a discrete case of  $n$  equally distributed time steps, formula (3.3) has the following form:

$$S(t_{i+1}) = S(t_i) \exp \left[ \left( r - \frac{1}{2} \sigma^2 \right) \Delta t + \sigma (W(t_i + \Delta t) - W(t_i)) \right].$$

Here  $\Delta t = T/n$ ,  $t_i = i\Delta t$ ,  $1 \leq i \leq n$ .

We consider two algorithms for the discretization of equation (3.2). The first one is known as the standard discretization algorithm. Its construction follows directly from the definition of  $W(t)$ . The second one is the alternative discretization algorithm which is based on the use of conditional distributions.

The standard discretization algorithm for stochastic differential equation (3.2) is defined by the relation:

$$W(t_i) = W(t_{i-1}) + \sqrt{\Delta t} Z_i, \quad 1 \leq i \leq n. \quad (3.4)$$

A terminal asset value  $S(T)$  is given by

$$S(T) = S_0 \exp \left[ \left( r - \frac{1}{2} \sigma^2 \right) T + \sigma \sqrt{\Delta t} (Z_1 + Z_2 + \dots + Z_n) \right].$$

In the standard discretization algorithm the evolution of an asset value is generated by normal variates with equal weights.

In the Brownian bridge discretization, the value of  $W(t_i)$  is generated from values of  $W(t_l), W(t_m), l \leq i \leq m$  at earlier and later time steps. Unlike the standard discretization, which generates  $W(t_{i+1})$  sequentially along the time horizon, the Brownian bridge discretization first generates the variable at the terminal point

$$W(T) = \sqrt{T} Z_1$$

and then it fills other points using already found values of  $W(t_i)$ . In the case of  $n = 2^p$ ,  $p$  is an integer number  $p > 0$ , subsequent values of  $W(t_i)$  are evaluated at the successive midpoints  $T/2, T/4, 3T/4, 3T/8, 5T/8, 7T/8, T/16$  and so on. In this case the Brownian bridge discretization scheme has a form (Sobol' 1962, Sobol' 1973):

$$\begin{aligned} W(T) &= \sqrt{T} Z_1, \\ W(T/2) &= \frac{1}{2} W(T) + \frac{1}{2} \sqrt{T} Z_2, \\ W(T/4) &= \frac{1}{2} W(T/2) + \frac{1}{2} \sqrt{T/2} Z_3, \\ W(3T/4) &= \frac{1}{2} (W(T/2) + W(T)) + \frac{1}{2} \sqrt{T/2} Z_4, \\ &\vdots \\ W((n-1)T/n) &= \frac{1}{2} (W((n-2)T/n) + W(T)) + \frac{1}{2} \sqrt{T/n} Z_n. \end{aligned} \quad (3.5)$$

The generalised Brownian bridge formula is given by

$$W(t_i) = (1 - \gamma) W(t_i) + \gamma W(t_m) + \sqrt{\gamma(1 - \gamma)(m - l)\Delta t} Z_i, \quad (3.5)$$

where  $\gamma = \frac{i-l}{m-l}$  (Morokoff, 1998). It can be seen from equation (3.5) that the variance of the stochastic part of the Brownian bridge formula  $\gamma(1 - \gamma)(m - l)\Delta t$ . It decreases rapidly at the successive levels of refinement. This variance is less than that in (3.4) as  $\gamma(1 - \gamma)(m - l) < 1$ . Therefore, the first few points contain the most of the variance. Both algorithms have the same variance, hence their MC convergence rates are the same but QMC algorithms have different efficiencies with the Brownian bridge algorithm having a much higher convergence rate (Caflisch et. al. 1997, Sobol' et. al. 2005).

### 3.2 Monte Carlo simulation of option pricing

In a risk-neutral environment, the value of European style options is the discounted value of its payoff:

$$C(K, T) = e^{-rT} E^Q [P(S(t), K)].$$

Here  $P(S(t), K)$  is a payoff function,  $K$  is the strike price,  $T$  is the time to maturity,  $r$  is a constant interest rate. A European call option provides the holder of the option with the right to buy the underlying asset by a certain date for a given price  $K$ . It has a payoff

$$P_E = \max(S_T - K, 0), \quad (3.6)$$

where  $S_T$  is the asset price at the maturity.

The payoff function for a geometric average Asian call option with asset prices in the average at  $n$  equally spaced time points is

$$P_A = \max(\bar{S} - K, 0), \quad (3.7)$$

where  $\bar{S}$  is a geometric average:  $\bar{S} = (\prod_{i=1}^n S_i)^{1/n}$ ,  $S_i$  is the asset price at time  $t_i = iT/n$ ,  $1 \leq i \leq n$ .

MC simulation of European style options is reduced to estimation of multidimensional integrals. For example, for the standard algorithm the price of a geometric average Asian call option can be written as the following  $n$ -dimensional integral:

$$C = e^{-rT} \int_{H^n} \max[0, \left( \prod_{i=1}^n S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)t_i + \sigma \sqrt{\frac{T}{n}} \sum_{j=1}^i \Phi^{-1}(u_j)\right] \right)^{1/n} - K] du_1 \dots du_n.$$

Here  $\Phi^{-1}(u)$  is an inverse cumulative function of a normal distribution,  $\{u_i\}$  are uniformly distributed in a unit hypercube  $H^n$  variates.

In a general case the MC method approximates the expectation of the derivative's payoff with a simple arithmetic average of payoffs taken over a finite number  $N$  of simulated price paths:

$$C_N = e^{-rT} \left[ \frac{1}{N} \sum_{i=1}^N P(S_0, S_1^{(i)}, \dots, S_T^{(i)}, K) \right]$$

For the case of a European call this formula has the following form (Jaekel 2002, Wilmott 2005):

$$C_N = \frac{1}{N} \sum_{i=1}^N C^{(i)} = e^{-rT} \left[ \frac{1}{N} \sum_{i=1}^N \max(S_T^{(i)} - K, 0) \right].$$

Global SA can be used for analysis of option pricing problems by quantifying the variation in the output variables to the variation of the input variables. This method is presented in the next section.

## 4 Global sensitivity analysis

Many practical problems deal with functions of a very complex structure. Global Sensitivity Analysis (SA) can provide information on the general structure of a function by quantifying the variation in the output variables to the variation of the input variables.

Consider an integrable function  $f(x)$  defined in the unit hypercube  $H^n$ . It can be expanded in the following form:

$$f(x) = f_0 + \sum_{s=1}^n \sum_{i_1 < \dots < i_s} f_{i_1 \dots i_s}(x_{i_1}, \dots, x_{i_s}) \quad (4.1)$$

Expansion (4.1) is a sum of  $2^n$  components. It can also be presented as

$$f(x) = f_0 + \sum_i f_i(x_i) + \sum_{i < j} f_{ij}(x_i, x_j) + \dots + f_{12 \dots n}(x_1, x_2, \dots, x_n).$$

Each of the components  $f_{i_1 \dots i_s}(x_{i_1}, \dots, x_{i_s})$  is a function of a unique subset of variables from  $x$ . The components  $f_i(x_i)$  are called first order terms,  $f_{ij}(x_i, x_j)$  - second order terms and so on.

It can be proven (Sobol', 1990) that the expansion (4.1) is unique if

$$\int_0^1 f_{i_1 \dots i_s}(x_{i_1}, \dots, x_{i_s}) dx_{i_k} = 0, \quad 1 \leq k \leq s, \quad (4.2)$$

in which case it is called a decomposition into summands of different dimensions. This decomposition is also known as the ANOVA (ANalysis Of VAriances) decomposition. The ANOVA decomposition is orthogonal, i.e. for any two subsets  $u \neq v$  an inner product

$$\int_0^1 f_u(x) f_v(x) dx = 0.$$

It follows from (4.1) and (4.2) that

$$\int_0^1 f(x) dx_{i_1} \dots dx_{i_n} = f_0,$$

$$\int_0^1 f(x) \prod_{k \neq i} dx_k = f_0 + f_i(x_i), \quad (4.3)$$

$$\int_0^1 f(x) \prod_{k \neq i, j} dx_k = f_0 + f_i(x_i) + f_j(x_j) + f_{i, j}(x_i, x_j)$$

and so on.

For square integrable functions, the variances of the terms in the ANOVA decomposition add up to the total variance of the function

$$\sigma^2 = \sum_{s=1}^n \sum_{i_1 < \dots < i_s} \sigma_{i_1 \dots i_s}^2, \quad (4.4)$$

where  $\sigma_{i_1 \dots i_s}^2 = \int_0^1 f_{i_1 \dots i_s}^2(x_{i_1}, \dots, x_{i_s}) dx_{i_1}, \dots, x_{i_s}$ .

Sobol' defined the global sensitivity indices as the ratios

$$S_{i_1 \dots i_s} = \sigma_{i_1 \dots i_s}^2 / \sigma^2.$$

All  $S_{i_1 \dots i_s}$  are non negative and add up to one

$$\sum_{s=1}^n \sum_{i_1 < \dots < i_s} S_{i_1 \dots i_s} = 1.$$

$S_{i_1 \dots i_s}$  can be viewed as a natural sensitivity measure of a set of variables  $x_{i_1}, \dots, x_{i_s}$ . It corresponds to a fraction of the total variance given by  $f_{i_1 \dots i_s}(x_{i_1}, \dots, x_{i_s})$ . For example,  $S_1$  is the main effect of a variable  $x_1$ ,  $S_{12}$  is a measure of interactions between  $x_1$  and  $x_2$  (i.e. that part of the total variance due to parameters  $x_1$  and  $x_2$  which cannot be explained by the sum of the effects of parameters  $x_1$  and  $x_2$  alone) and so on. An analysis of  $S_{i_1 \dots i_s}$  provides sufficient information about the structure of a corresponding function. For functions of an additive structure, only the low order sensitivity indices are important. In an extreme case in which there is no interaction among the input variables

$$f(x) = f_0 + \sum_i f_i(x_i)$$



all higher order sensitivity indices are equal to zero and

$$\sum_{s=1}^n S_i = 1.$$

In the general case all global sensitivity indices can be important. Their straightforward calculation using the ANOVA decomposition would result in  $2^n$  integral evaluations of the summands  $f_{i_1 \dots i_s}(x_{i_1}, \dots, x_{i_s})$  using (4.3) and  $2^n$  integral evaluations for calculations of  $\sigma_{i_1 \dots i_s}^2$  (4.4). For high dimensional problems such an approach is impractical. For this reason Sobol' (1990) introduced sensitivity indices for subsets of variables and the total sensitivity indices. Consider two complementary subsets of variables  $y$  and  $z$ :

$$x = (y, z).$$

Let  $y = (x_{i_1}, \dots, x_{i_m})$ ,  $1 \leq i_1 < \dots < i_m \leq n$ ,  $K = (i_1, \dots, i_m)$ . The variance corresponding to  $y$  is defined as

$$\sigma_y^2 = \sum_{s=1}^m \sum_{(i_1 < \dots < i_s) \in K} \sigma_{i_1 \dots i_s}^2.$$

$\sigma_y^2$  includes all partial variances  $\sigma_{i_1}^2, \sigma_{i_2}^2, \dots, \sigma_{i_1 \dots i_s}^2$  such that their subsets of indices  $(i_1, \dots, i_s) \in K$ . The total variance  $(\sigma_y^{tot})^2$  is defined as

$$(\sigma_y^{tot})^2 = \sigma^2 - \sigma_z^2$$

$(\sigma_y^{tot})^2$  consists of all  $\sigma_{i_1 \dots i_s}^2$  such that at least one index  $i_p \in K$  while the remaining indices can belong to the complimentary to  $K$  set  $\bar{K}$ . The corresponding global sensitivity indices are defined as

$$S_y = \sigma_y^2 / \sigma^2,$$

$$S_y^{tot} = \sigma_y^{tot} / \sigma^2.$$

$S_y^{tot} = 1 - S_z, S_y^{tot} - S_y$  accounts for all interactions between  $y$  and  $z$ .

One of the most important results obtained by Sobol' is an effective way of computing sensitivity indices (see Sobol' 2001 for details).

The important indices in practice are  $S_i$  and  $S_i^{tot}$ . Their knowledge in most cases provides sufficient information to determine the sensitivity of the analyzed function to individual input variables. The use of  $S_i$  and  $S_i^{tot}$  reduces the number of index calculations from  $O(2^n)$  to just  $O(2n)$ . Extreme cases are

- $S_i^{tot} = 0$  means that  $f(x)$  does not depend on  $x_i$  (in this case  $S_i$  is also equal to 0);
- $S_i = 1$  means that  $f(x)$  depends only on  $x_i$  (in this case  $S_i^{tot}$  is also equal to 1);
- $S_i = S_i^{tot}$  corresponds to the absence of interactions between variable  $x_i$  and other variables.

The notion of the "effective dimension" was introduced Caflisch et al (1997). Let  $|u|$  be a cardinality of a set  $u$ . Then the effective dimension of

$f(x)$  in the superposition sense is the smallest integer  $d_s$  such that

$$\sum_{0 < |u| < d_s} S_u \geq 0.99. \quad (4.5)$$

The threshold 0.99 is arbitrary. Condition (4.5) means that the function  $f(x)$  is almost a sum of  $d_s$  - dimensional functions. The effective dimension in the truncation sense  $d_T$  is defined as

$$\sum_{u \subseteq \{1, 2, \dots, d_T\}} S_u \geq 0.99 \quad (4.6)$$

The value  $d_s$  does not depend on the order in which the input variables are sampled, while  $d_T$  does. It was suggested that the efficiency of QMC methods on high dimensional problems can be attributed to the low effective dimension of the integrand (in one or both of the senses), although no formal proof was given. By reducing the effective dimension, a higher efficiency of QMC integration can be achieved. One example of such an approach is a simulation driven by Brownian motion. By changing the order in which the variables are sampled from LDS the effective dimension can be reduced and thus the accuracy can be significantly improved. Unfortunately, a straightforward evaluation of the effective dimensions from their definitions (4.5), (4.6) is not practical in the general case as it would require the calculation of all  $2^n$  components  $S_u$ . Owen introduced the dimension distribution for a square integrable function (Owen, 2003). The effective dimension can be defined through a quantile of the dimension distribution but such quantiles are still hard to estimate. Global SA offers a general practical way to predict the efficiency of QMC methods.

## 5 Numerical results

In this section we present the numerical results from simulations of prices of European and Asian call options. The simulations are performed to show the difference between the convergence of MC and QMC methods with the standard and the Brownian Bridge discretization schemes.

### 5.1 Pricing of European call

The following parameters were used for simulation:  $S_0 = 100$ ,  $K = 100$ ,  $r = 0.05$ ,  $\sigma = 0.2$ ,  $T = 0.5$ . An exact Black-Scholes value of the option price is 6.89. The number of time steps used in the simulation,  $n = 32$ . The European call option follows a path-independent process and we only need to simulate the terminal asset price. However, here we simulate the entire price path using a discrete approximation to demonstrate the difference in the performance of different techniques.

Fig. 1 shows the results of simulation of a European call option price versus the number of paths  $N$  obtained using MC and QMC method with standard and Brownian Bridge discretizations. The Mersenne Twister generator, which is considered to be one of the most efficient uniform random number generators was used for MC simulation (Mersenne Twister, 2007). The Sobol' sequence generator SobolSeq was used for QMC simulations (SobolSeq, 2007). Results of MC simulation show that simulated solution slowly converges to the exact solution but the convergence curve is highly oscillating. In contrast, QMC convergence is practically monotonic

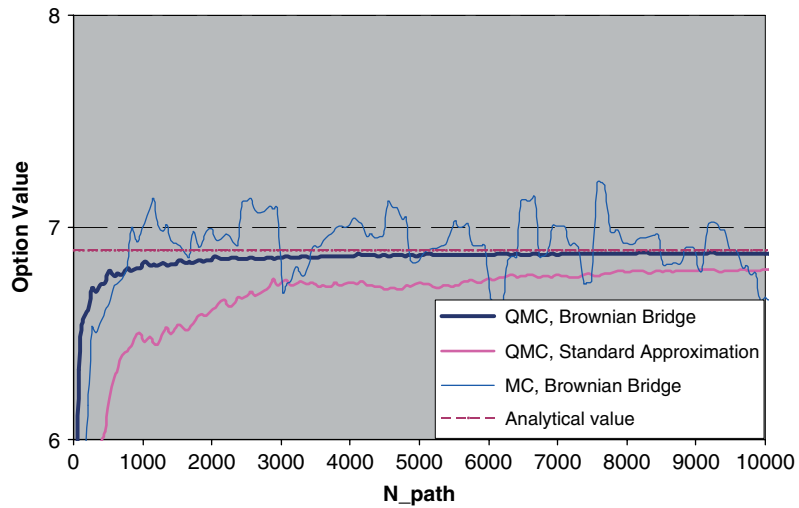


Fig. 1 European call price versus the number of paths obtained using MC and QMC method with standard and Brownian Bridge discretizations.

which makes on-line error approximation possible. In the case of QMC method with the Brownian bridge construction convergence is much faster than that for the standard discretization; however, the type of discretization does not effect the convergence in the case of MC method.

Fig. 2 shows the root mean square error (RMSE) versus the number of paths for same methods. To reduce the scatter in the error estimation the values of root mean square errors

$$\varepsilon = \left( \frac{1}{L} \sum_{l=1}^L (C - C_N^{(l)})^2 \right)^{1/2},$$

were averaged over  $L = 50$  independent runs. Here  $C_N^{(l)}$  is an estimated option value on  $l$ -th run for  $N$ -paths replications. For the MC method all runs were statistically independent. For QMC integration for each run a different part of the Sobol' sequence was used.

The results show the superior performance of the QMC approach with the Brownian Bridge discretization. The convergence rate for the Brownian Bridge discretization decreases as  $1/N^{0.88}$ , while for the standard discretization this rate is only  $1/N^{0.66}$  and for MC it is  $\sim 1/N^{0.49}$ , which is very close to a theoretically predicted limit (2.4).

## 5.2 Pricing of Asian call with geometric averaging.

The Asian call option follows a path-dependent process. Two sets of parameters were used for simulation:

A ( low dimensional case ):  $S_0 = 100$ ,  $K = 100$ ,  $r = 0.05$ ,  $\sigma = 0.2$ ,  $t = 0.5$ , number of discrete time steps  $n = 32$ . Exact Black-Scholes Value of the option price is 3.84.

B ( high dimensional case ):  $S_0 = 100$ ,  $K = 100$ ,  $r = 0.05$ ,  $\sigma = 0.2$ ,  $T = 1.0$ , number of discrete time steps  $n = 252$ . Exact Black-Scholes Value of the option price is 5.56.

Fig. 3 shows the results of simulation of an Asian call price versus the number of paths obtained using MC and QMC method with standard and Brownian Bridge discretizations ( case A ). Qualitatively, the results

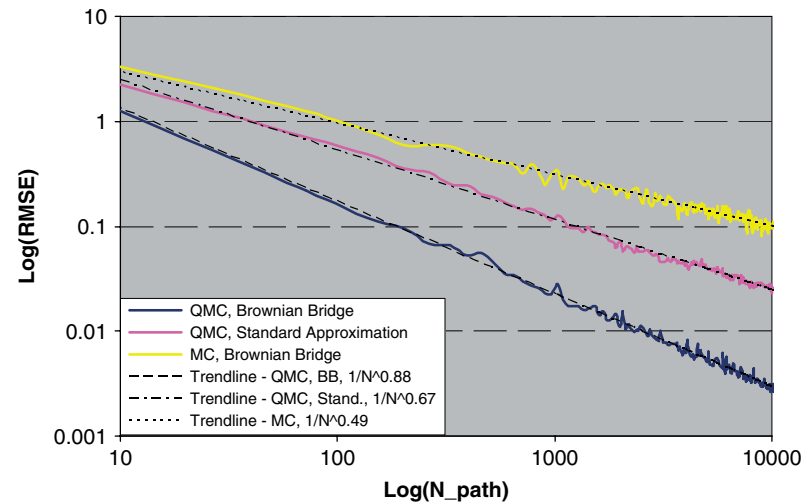


Fig. 2 Log-log plot of the root mean square error versus the number of paths for QMC method with standard and Brownian Bridge discretizations and for the MC method.

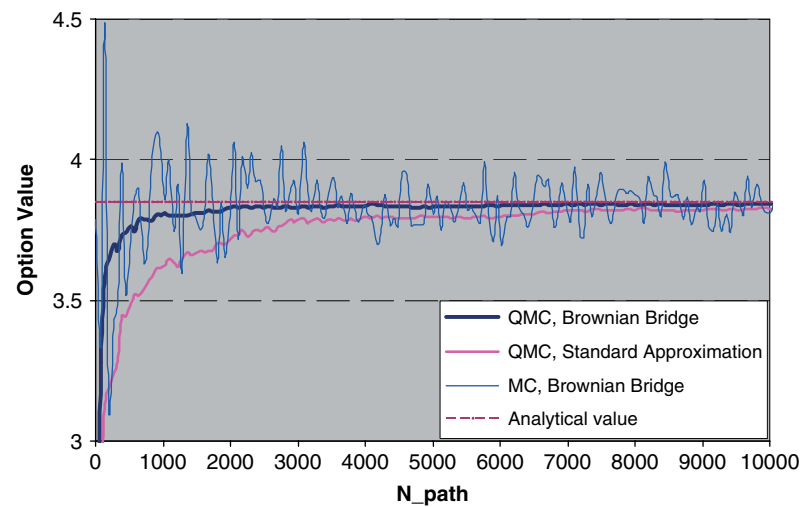


Fig. 3 Asian call price versus the number of paths obtained using MC and QMC method with standard and Brownian Bridge discretizations. Case A ( $n = 32$ ).

are similar to the case of a European call, that is the Brownian bridge construction provides much better convergence when using Sobol' sequence sampling than the standard discretization or the MC method.

Fig. 4 shows RMSE versus the number of paths. The convergence rate of the QMC approach based on Sobol' sequences with the Brownian Bridge discretization follows  $1/N^{0.85}$ , for the standard discretization this rate is  $1/N^{0.7}$  and for MC it is only  $\sim 1/N^{0.49}$ .

To investigate the dependence of MC and QMC methods on dimensionality, we considered case B with 252 time steps. Fig. 5 shows RMSE versus the number of paths for this case. Results show that the convergence rate for the Brownian Bridge discretization does not practically

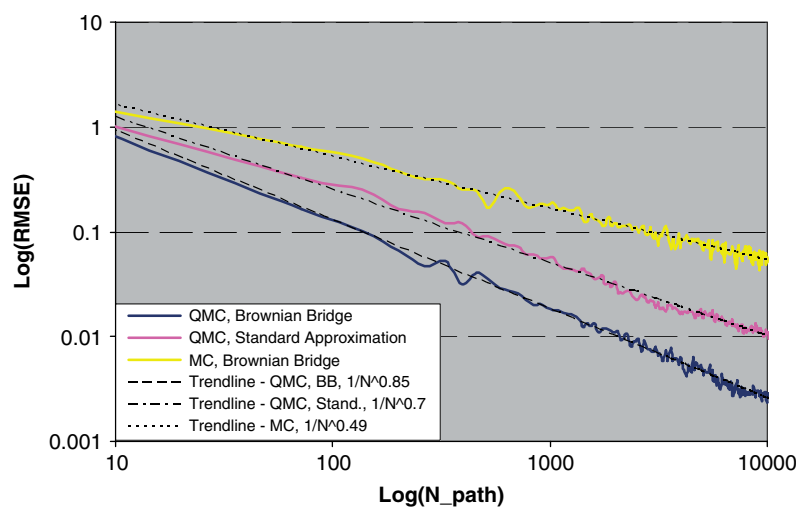


Fig. 4 Log-log plot of the root mean square error versus the number of paths for QMC and MC methods. Asian call, case A ( $n = 32$ ).

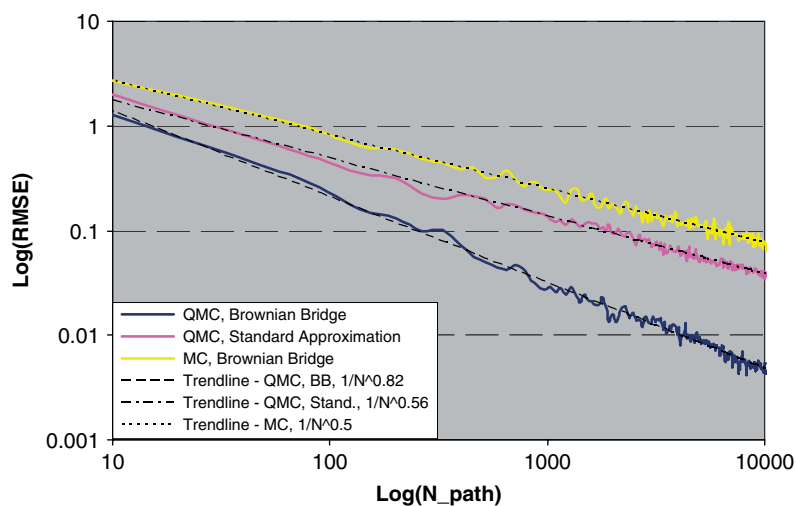


Fig. 5 Log-log plot of the root mean square error versus the number of paths for QMC and MC methods. Asian call, case B ( $n = 252$ ).

depend on dimensionality, while for the standard discretization the convergence rate decreases from  $1/N^{0.7}$  ( $n = 32$ ) to  $1/N^{0.56}$  ( $n = 252$ ) and it becomes close to the MC convergence rate  $1/N^{0.5}$ .

An important factor in the comparison of methods is the overall computation time. It can be seen from the results that applying QMC method with standard discretization reduces the number of required simulations by 10 times compared to MC (case A). Combining the effective dimension reduction technique achieved via the Brownian bridge discretization, together with QMC sampling cuts the computational effort further to one-hundredth of that required when using MC. It is important to note, that the Sobol' sequence numbers are generated considerably faster than many other LDS and even faster than generation of pseudo-random numbers by known generators.

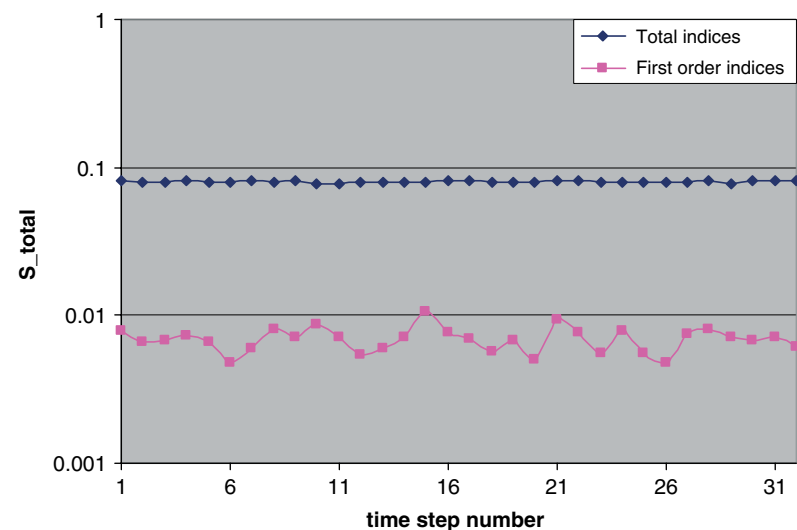


Fig. 8 First order  $S_i$  and total sensitivity indices  $S_i^T$  versus time step  $i$ . European call option, standard discretization,  $n = 32$ .

### 5.3 Global Sensitivity Analysis of the standard discretization and the Brownian Bridge construction.

To understand why the standard and Brownian bridge discretizations have different efficiencies, we applied global SA to payoff functions (3.6), (3.7), which we write explicitly as functions of the set of normal variables  $\{Z_i\}$ . For the case of a European call option the payoff function is

$$P_E(\{Z_i\}) = \max(S_T(\{Z_i\}) - K, 0)$$

Fig. 8 shows values of first order sensitivity indices  $S_i$  and total sensitivity indices  $S_i^T$  versus time step  $i$  for standard discretization with  $n = 32$ . The points are linked for clarity by broken lines. Slight oscillations in the values are the result of MC approximations of sensitivity indices values. It can be seen that all variables  $\{Z_i\}$  are equally important. The ratio  $S_i/S_i^T$  is close to 0.01 which means that interactions play important role and the effective dimension for this type of discretization is close to the real dimension  $n$ .

The situation is very different for the Brownian bridge discretization:  $S_1 = S_1^T = 1$  and  $S_i = S_i^T = 0$ ,  $1 < i \leq n$ . It means that the effective dimension in the truncation sense for this discretization  $d_T$  is equal to 1.

For the case of an Asian call option we analyzed the payoff functions is

$$P_A(\{Z_i\}) = \max(\bar{S}(\{Z_i\}) - K, 0)$$

Figures 9 and 10 show values of total sensitivity indices versus time step index  $i$  for an Asian call option with  $n = 32$  discretely sampled time steps. The points are linked for clarity by broken lines. For the standard discretization total sensitivity indices slowly decrease with the increase of the time step index  $i$ . For the Brownian bridge discretization the total sensitivity indices of the first few variables are much larger than those

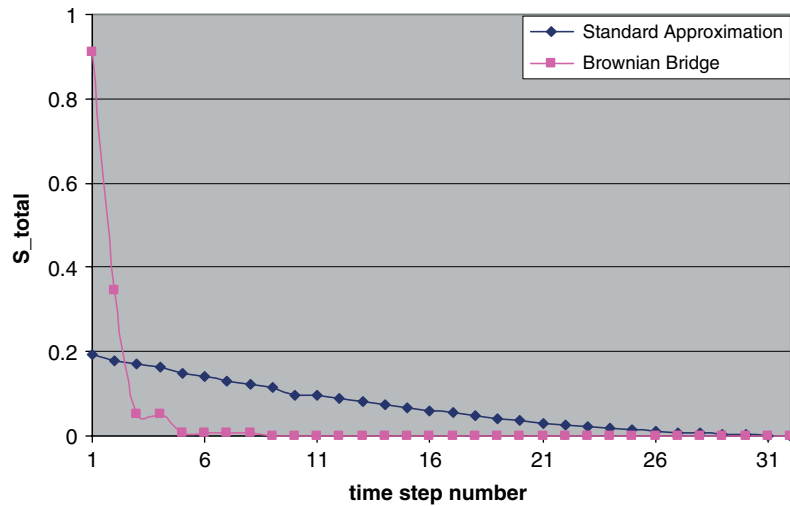


Fig. 9 Total sensitivity indices  $S_i^T$  versus time step  $i$ . Asian call,  $n = 32$ .

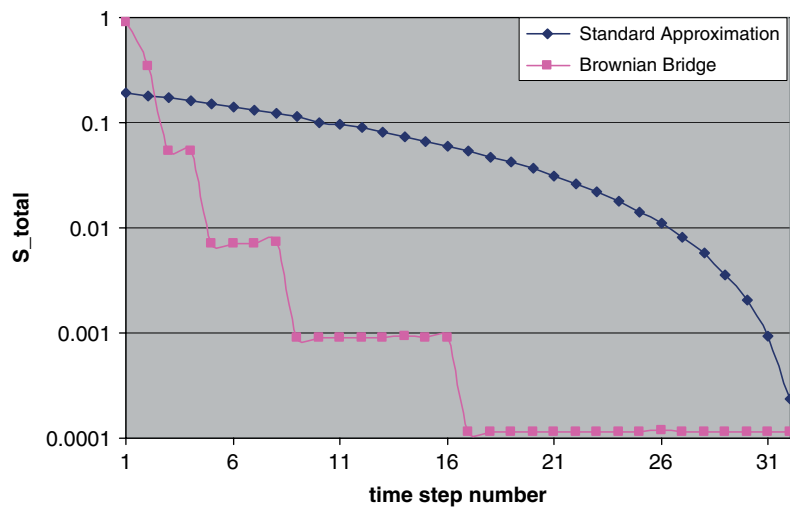


Fig. 10 Log of total sensitivity indices  $S_i^T$  versus time step number  $i$ . Asian call,  $n = 32$ .

of the subsequent variables. They also decrease more rapidly than total sensitivity indices for the standard discretization. In accordance with (3.4) the first two total sensitivity indices for the Brownian bridge are considerably larger than those for the standard method. It results in particular in the much higher value of the sum of the first order sensitivity indices for the Brownian bridge discretization than that for the standard discretization (Table 1).

Results presented in Table 1 also show that the contribution of the first order terms in the ANOVA representation for the standard discretization is small and it decreases with the increase of the number of steps  $n$ . As a result the importance of higher order interactions grows with  $n$ . In contrast, for the Brownian Bridge discretization the sum of the first order sensitivity indices is practically independent of  $n$ . The ratio  $S_1 / \sum_{i=1}^n S_i = 0.98$ , which

**Table 1. Sum of the first order sensitivity indices for the standard and Brownian Bridge discretizations for various  $n$**

$n$	Option Value	$\sum_i S_i$ Stand.	$\sum_i S_i$ BB
8	4.13	0.102	0.41
16	3.94	0.042	0.38
32	3.84	0.022	0.37

means that for the Brownian Bridge discretization the effective dimension in the truncation sense is close to 1. In the superposition sense it is larger than 1 as interactions between variables are somewhat important:  $S_1/S_1^T = 0.62$ ,  $S_i/S_i^T \approx 0.15$ ,  $1 < i \leq n$ .

The initial coordinates of LDS are much better distributed than the later high dimensional coordinates (Sobol' et al., 1992, Caflisch et al., 1997). As follows from global SA for the Brownian bridge discretization the low index variables are much more important than higher index variables. The Brownian bridge discretization uses low well distributed coordinates from each  $n$ -dimensional LDS vector point to determine most of the structure of a path and reserves the later coordinates to fill in fine details. In other words, well distributed coordinates are used for important variables and higher not so well distributed coordinates are used for far less important variables. It results in a significantly improved accuracy of QMC integration. In contrast, the standard construction does not account for the specifics of LDSs distribution properties.

## 6 Conclusions

The Brownian bridge discretisation of the Brownian path results in significant improvement of the accuracy of QMC especially when the number of time steps  $n$  is large. Global SA offers an efficient and general approach for analysis and reduction of problem complexity. It reveals that the variance of the samples generated for the Brownian path slowly decreases with time step index for the standard discretisation for the case of an Asian call and it is constant for the case of a European call. The higher order interactions in the ANOVA decomposition of payoff functions are very important. Therefore, the effective dimensions for this discretisation is close to the real dimension.

For the Brownian bridge discretisation the sensitivity indices of the first few variables are much larger than those of the subsequent variables. Application of the Brownian bridge discretization greatly reduces the effective dimension in the truncation sense and consequently increases the efficiency of QMC. Its efficiency does not depend on the problem dimensionality. Although the standard discretisation with QMC sampling is superior to MC, the convergence rate of the QMC method is much lower than that of the Brownian bridge discretisation and it decreases as dimensionality grows.

Although for easiness of comparison we considered options which have closed-form solutions, the presented techniques can be applied for any path dependent options.



## Acknowledgement

We express our gratitude to Paul Wilmott for helpful discussions and his interest in this work. Authors gratefully acknowledge the financial support by the EPSRC grant EP/D506743/1.

## REFERENCES

- R.E. Caflisch, W. Morokoff, A.B. Owen. Valuation of Mortgage Backed Securities using Brownian Bridges to Reduce Effective Dimension. *The Journal of Computational Finance*. 1 (1997) 27–46.
- F. Campolongo, J. Cariboni, and W. Schoutens. The importance of jumps in pricing European options *Reliability Engineering & System Safety* V 91, Issues 10–11 (2006), 1148–1154.
- P. Jaeckel. *Monte Carlo Methods in Finance*. John Wiley & Sons, London, 2002.
- C. Lemieux, A. B. Owen. Quasi-regression and the relative importance of the ANOVA components of a function, in *Monte Carlo and quasi-Monte Carlo methods, 2000* (Hong Kong), Springer, Berlin, 2002.
- S. Kucherenko, W. Mauntz. Application of global sensitivity indexes for measuring the effectiveness of quasi-Monte Carlo method. Submitted to *Monte Carlo Methods and Simulation* (2007).
- Mersenne Twister. <http://www.math.sci.hiroshima-u.ac.jp/~m-mat/MT/emt.html>, 2007.
- B. Moskowitz, R.E. Caflisch. Smoothness and Dimension Reduction in Quasi-Monte-Carlo Methods, *Mathl. Comput. Modelling*, 23 (8/9) (1996) 37–54.
- W. J. Morokoff. Generating Quasi-Random Paths for Stochastic Processes, *SIAM Review*, v. 40, 4, (1998) 765–788.
- A. B. Owen. The dimension distribution and quadrature test functions, *Statistica Sinica* 13 (2003) 1–17.
- I. M. Sobol' Computation of definite intergrals, in "The Monte Carlo Method", ed. Yu. A. Shreider, Pergamon Press, 1966, translated from "The method of statistical trials", Fizmatgiz, Moscow 1962 (in Russian).
- I. M. Sobol'. Numerical methods Monte Carlo. Moscow, 1973 ( in Russian ).
- I. M. Sobol'. Sensitivity estimates for nonlinear mathematical models. *Matematicheskoe Modelirovanie* 2 (1) (1990) 112–118 (in Russian), translated in I.M. Sobol', Sensitivity estimates for nonlinear mathematical models. *Mathematicai Modeling and Computational Experiment*. 1 (1993) 407–414.
- I. M. Sobol', S. Kucherenko. On global sensitivity analysis of quasi-Monte Carlo algorithms. *Monte Carlo Methods and Simulation*, 11, 1, (2005) 1–9.
- I. M. Sobol', V.I.Turchaninov, Yu.L.Levitan, B.V.Shukhman. Quasirandom sequence generators. Preprint. Keldysh Inst. Appl. Math, Moscow, 1992.
- I. M. Sobol'. Global sensitivity indices for nonlinear mathematical models and their Monte Carlo estimates. *Mathematics and computer in simulation*, v.55, (2001) 271–280.
- I. M. Sobol', S. Kucherenko. On global sensitivity analysis of quasi-Monte Carlo algorithms. *Monte Carlo Methods and Simulation*, 11, 1, 1–9, 2005.
- I.M. Sobol', S. Kucherenko. Global Sensitivity Indices for Nonlinear Mathematical Models. *Review, Wilmott*, 1 (2005), 56–61.
- SobolSeq. <http://www.broda.co.uk>, 2007.
- P. Wilmott. *Paul Wilmott on Quantitative Finance*, John Wiley & Sons, London, 2006.