Comparison of BRODA's and Joe&Kuo' Sobol' sequence generators

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https://www.broda.co.uk/

Outline

Construction of Sobol' sequences:

1. Joe and Kuo's generator: better 2D projections
2. BRODA's generator: Propeties A and A'

Comparison on the following test cases:

1. Discrepancies
2. Integration
3. Spurious variance component
4. Pricing of at the money call
5. Asian Option pricing
6. CVA and CVA Sensitivities

Definitions. I

Definition 1. Dyadic intervals. Let m be a nonnegative integer. Dyadic intervals with length 2^{-m} are obtained dividing the unit interval $0 \le z \le 1$ into 2^m equal intervals.

Definition 2. Dyadic boxes. A dyadic box Π in the d-dimensional unit hypercube H^d is a product of d dyadic intervals.

Thus a given set of d nonnegative integers $(m_1,\ldots,m_d)$ defines a decomposition of H^d into a sum of 2^m dyadic boxes with volumes 2^{-m} where $m = m_1 + \ldots + m_d$.

Definition 3. P_τ-nets. Consider two integers $\nu > \tau \ge 0$. A point set consisting of $N = 2^\nu$ points in H^d is called a P_τ-net if each dyadic box with volume $2^\tau/N$ contains exactly 2^τ points of the net.

Smaller values of τ imply a more uniform distribution of the points of the P_τ-net.

Definitions. II

Definition 4. Let $x_0, x_1, x_2, \ldots$ be an infinite sequence of points in H^d. A subset of points x_i with indices i satisfying inequalities $(k - 1)2^p \le i < k2^p$ with arbitrary positive integers k and p is called a dyadic section of the sequence. 2^p is the length of the section.

Definition 5. An infinite sequence of points $x_0, x_1, x_2, \ldots$ in H^d is called an LP_τ-sequence if all its dyadic sections with lengths exceeding 2^τ are P_τ-nets.

LP_0-sequences exist only in H^1 and H^2.

Definition 6. Niederreiter (1988) defined P_τ-nets and LP_τ-sequences as "(t,m,d)-nets in base 2" and "(t,d)-sequences in base 2", respectively.

Here $\nu = m$, $\tau = t$. They are known as Sobol' sequences

Construction of Sobol' Sequence. I

Integer n written in base 2: $n = (\cdots b_4 b_3 b_2 b_1 b_0)_2$

In the decimal system: $n = \sum_{j=0}^{L-1} b_j 2^j\ ,\ b_j \in [0,1]$

$u_n = (b_0 b_1 b_2 b_3 \ldots b_{L-1})_2$ – in reversed order

Construct vector $g_n^j = V^j u_n$, V^j permutation matrix for dimension j.

Decimal number x_n^j is n-th element of the sequence for dimension j:

$$x_n^j = \sum_{l=1}^{L} g_l^j 2^{-l}$$

Van der Corput sequence $h(n;2)$, if $V^j = I$: $h(n;2) = \sum_{l=0}^{L-1} b_j 2^{-l-1}$

Sobol' sequence – A permutation of Van der Corput sequence in base 2.

In practice:

1. Construct direction numbers:

$$v^j = (0.v_1^j v_2^j v_3^j \ldots v_L^j)_2$$

2. The Sobol' number $y_n^j = b_0 v_1^j \oplus b_1 v_2^j \oplus \ldots \oplus b_{L-1} v_L^j$,

where $\oplus$ is an addition modulo 2

Construction of Sobol' Sequence. II

3. Convert integer $y_n^j(i)$ to a uniform variate:

$x_n^j = y_n^j / 2^j$

Generation of direction numbers:

Let $P_j = x^{s_j} + a_{1,j} x^{s_j - 1} + \cdots + a_{s_j - 1} x + 1,$

be a set of different primitive polynomials, $j = 1, \ldots, d$.

Define a set of integers $m_{i,j}$ at any $j \geq 2,\ i > s_j,$

where s_j is an order of P_j corresponding to dimension j by recurrence:

$$m_{i,j} = 2a_{1,j} m_{i-1,j} \oplus 2^2 a_{2,j} m_{i-2,j} \oplus \ldots \oplus 2^{s_j - 1} a_{s_j - 1, j} m_{i - s_j + 1, j} \oplus 2^{s_j} m_{i - s_j, j} \oplus m_{i - s_j, j}$$

Then $v_i^j = \dfrac{m_{i,j}}{2^i}$.

Initial values $m_{1,j}, m_{2,j}, \ldots, m_{s_j,j}$ can be chosen arbitrarily provided that conditions

$m_{k,j} < 2^k$ and $m_{k,j}$ is odd are satisfied.

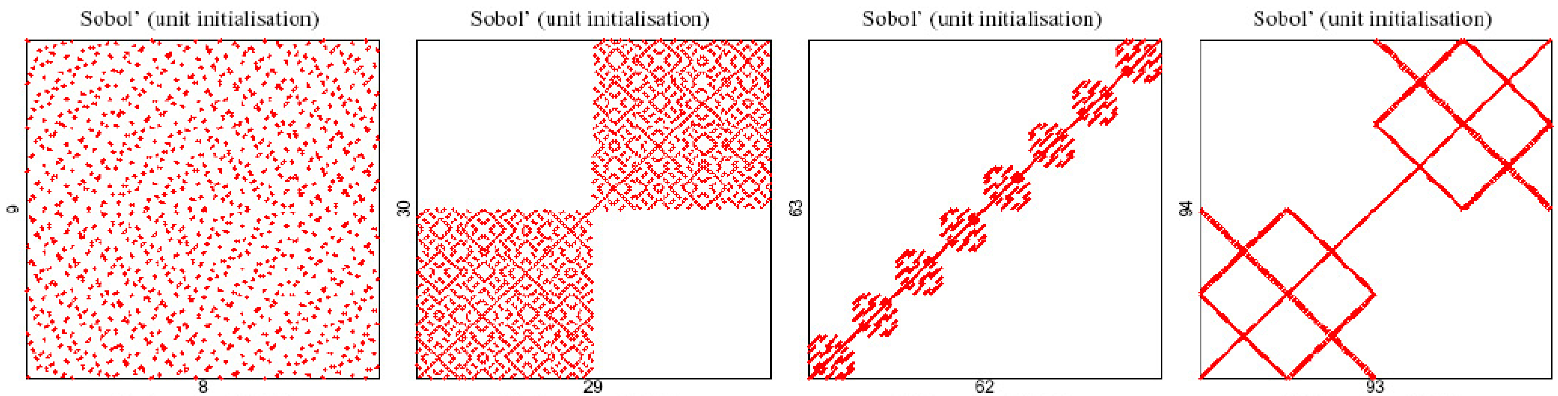
Therefore, it is possible to construct different Sobol sequences for the fixed dimension d.

Joe and Kuo: Constructing Sobol' sequences with better 2D projections

Recall: The smaller the *t*-value is, the more uniformly distributed points are.
In higher dimensions as *d* increases the smallest possible values of *t* increase as well:

$$t = \sum_{j=1}^{d} (s_j - 1)$$

-> use primitive polynomials P_j with as low as possible s_j.
Badly initialized Sobol' sequences can have poor 2D projections at low number of *N*



Ref: Peter Jackel, Monte Carlo Methods in Finance, John Wiley & Sons, 2002

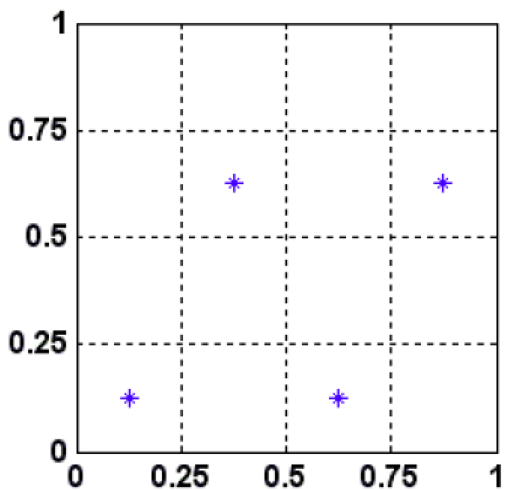
Joe and Kuo (2008, 2021) provided a set of direction numbers for Sobol' (t; d)-sequences by optimizing *t* values of 2D projections for d <=21201.

[1] S. Joe and F. Y. Kuo. Remark on algorithm 659: Implementing Sobol's quasirandom sequence generator. ACM Trans. Math. Software, 29:49–57, 2003.
[2] S. Joe and F. Y. Kuo. Constructing Sobol sequences with better two dimensional projections. SIAM J. Scientific Comp., 30:2635–2654, 2008.

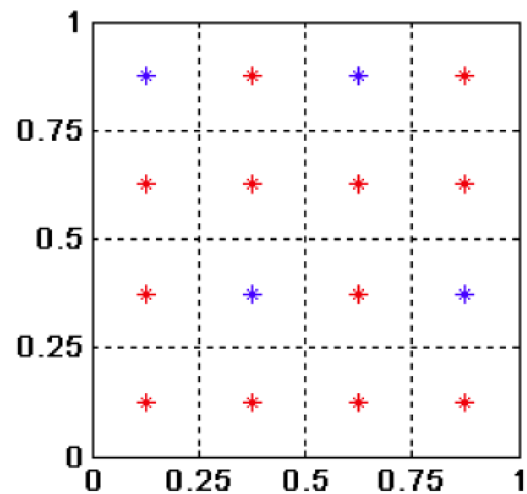
BRODA: Constructing Sobol' sequences with Properties A and A'

Property A. Consider *d*-dimensional hypercube which is cut by plains x_j=1/2 into 2^d subcubes. Sequence of Sobol points satisfies Property A, if after dividing the sequence into blocks of 2^d points, each one of the points in any one block belongs to a different subcube

Property A'. Consider *d*-dimensional hypercube which is cut by plains x_j=*k*/4, j=1,..,*n*, *k*=1,2,3 into 4^d subcubes. Sequence of Sobol points satisfies Property A', if after dividing the sequence into blocks of 4^d points, each one of the points in any one block belongs to a different subcube



Property A



Property A'

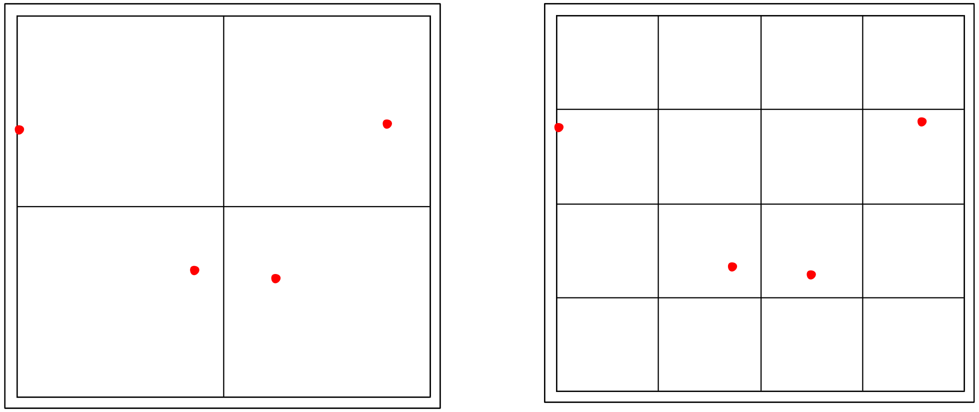
BRODA: Property A for d <=65536, Property A' for 5 adjacent dimensions

I. Sobol', D. Asotsky, A. Kreinin, S. Kucherenko. Construction and Comparison of High-Dimensional Sobol' Generators, 2011, Wilmott Journal, Nov, 64-79

Distributions of 4 points in 2D

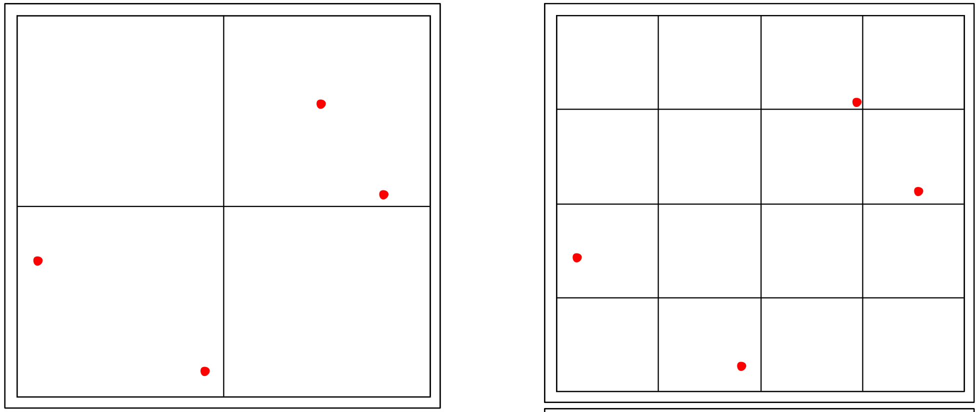
Property A

MC ->



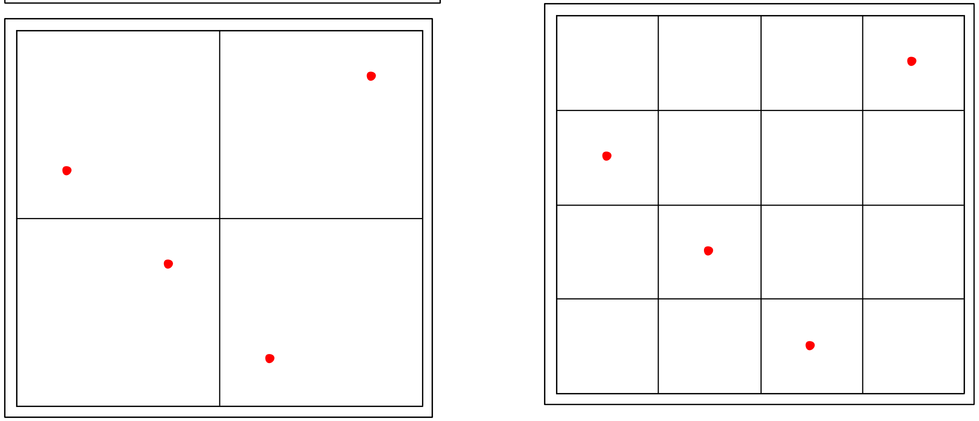
No

LHS ->



No

Sobol' ->



Yes

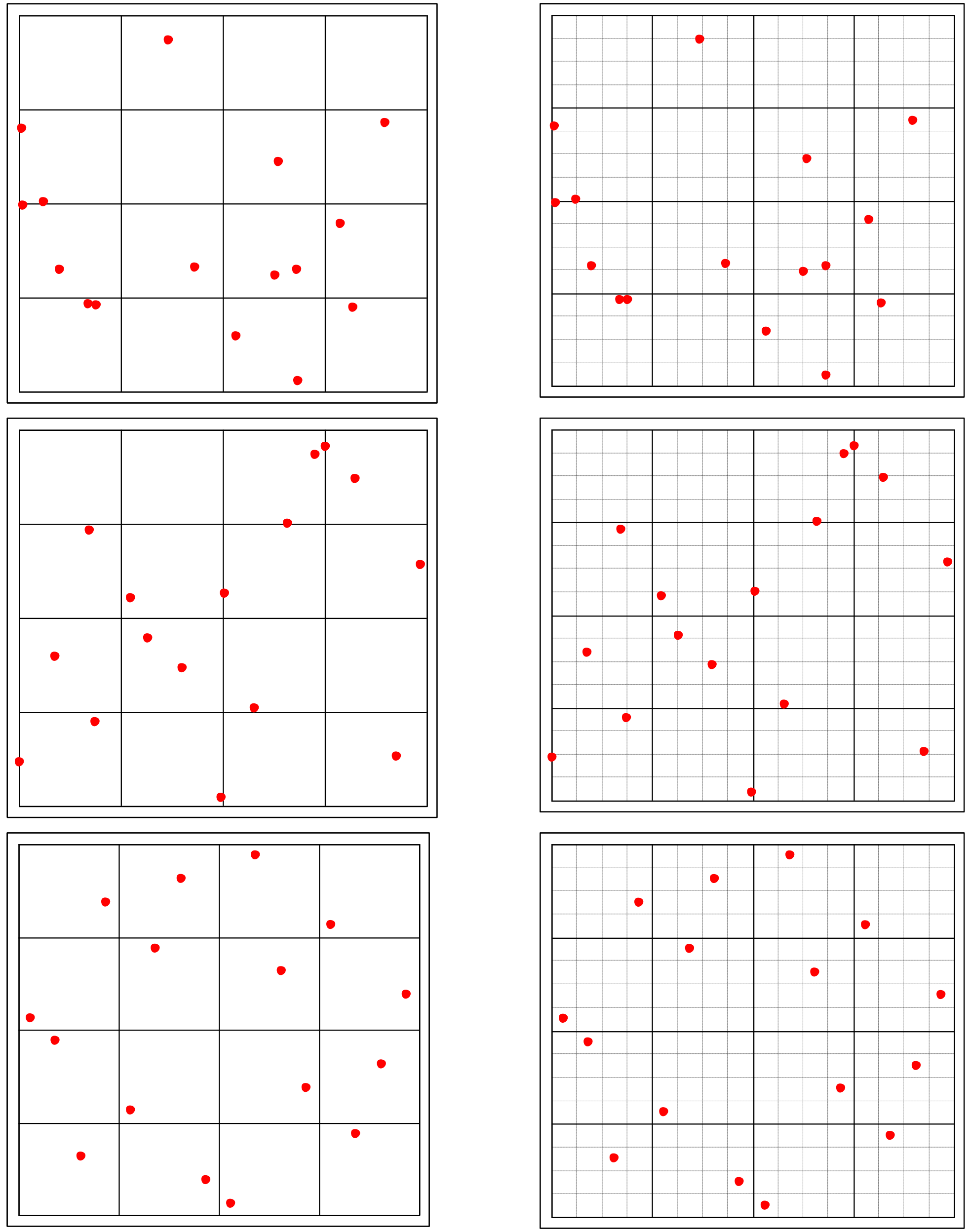
Distributions of 16 points in 2D

Property A'

MC -> No

LHS -> No

Sobol' -> Yes



BRODA: Constructing Sobol' sequences with Properties A and A'

Theorem 1. The d-dimensional Sobol sequence possesses Property A if and only if

$$det(V_d) = 1(mod\ \ 2),$$

where V_d is the dxd binary matrix defined by

$$V_d = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & v_{2,2,1} & v_{3,2,1} & \cdots & v_{d,2,1} \\ 1 & v_{2,3,1} & v_{3,3,1} & \cdots & v_{d,3,1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & v_{2,d,1} & v_{3,d,1} & \cdots & v_{d,d,1} \end{pmatrix}$$

with $v_{k,j,1}$ denoting the first digit after the binary point of the k-th direction number for dimension j $(v_{k,j} = (0.v_{k,j,1}v_{k,j,2} \dots)_2)$.

Theorem 2. The d-dimensional $Sobol'$ sequence possesses Property A' if and only if

$$det(U_d) = 1(mod\ \ 2),$$

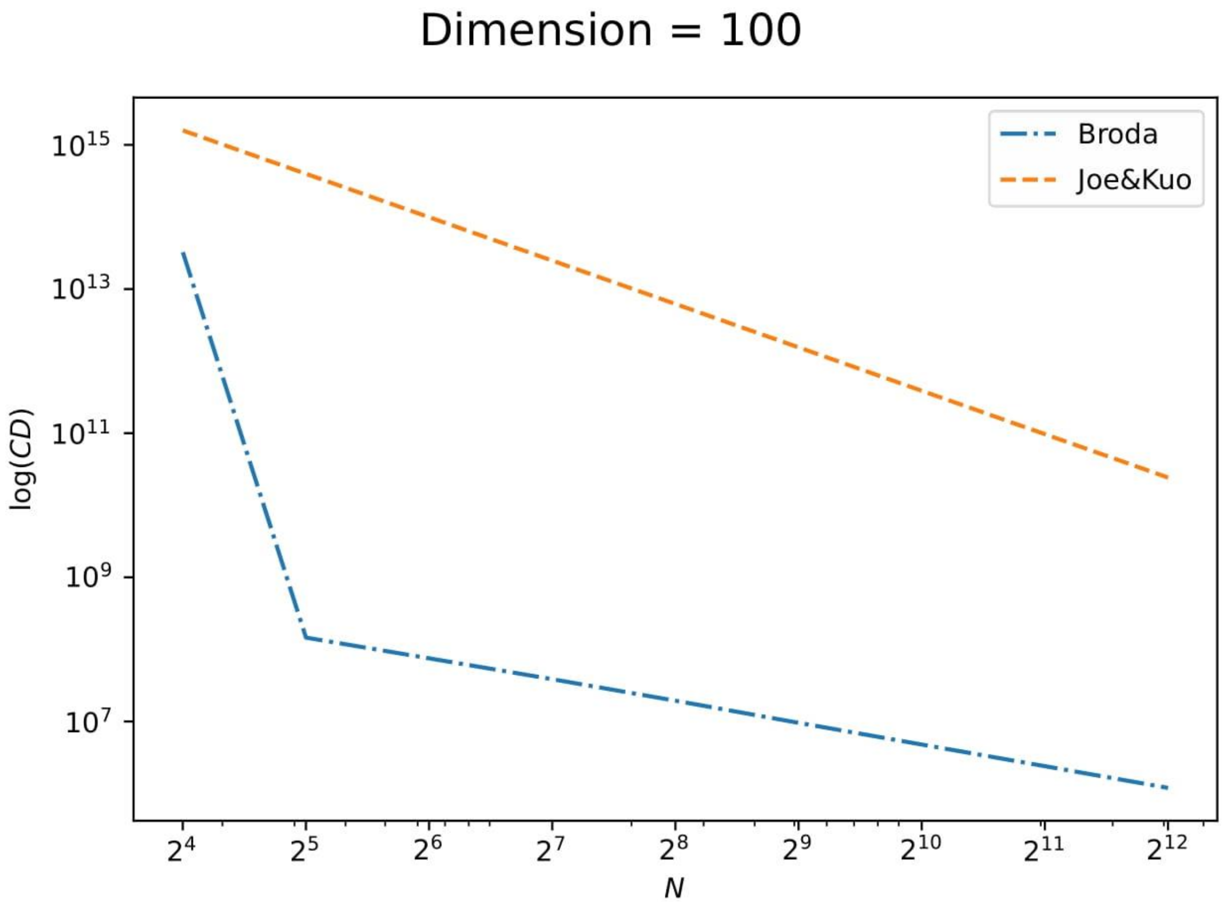
where U_d is the $2dx2d$ binary matrix defined by

$$U_d = \begin{pmatrix} v_{1,1,1} & v_{1,1,2} & v_{2,1,1} & v_{2,1,2} & \cdots & v_{d,1,1} & v_{d,1,2} \\ v_{1,2,1} & v_{1,2,2} & v_{2,2,1} & v_{2,2,2} & \cdots & v_{d,2,1} & v_{d,2,2} \\ v_{1,3,1} & v_{1,3,2} & v_{2,3,1} & v_{2,3,2} & \cdots & v_{d,3,1} & v_{d,3,2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{1,2d,1} & v_{1,2d,2} & v_{2,2d,1} & v_{2,2d,2} & \cdots & v_{d,2d,1} & v_{d,2d,2} \end{pmatrix}$$

with $v_{k,j,i}$ denoting the i-th digit after the binary point of the k-th direction number for dimension j $(v_{k,j} = (0.v_{k,j,1}v_{k,j,2} \dots)_2$

Centered Discrepancy (CD)

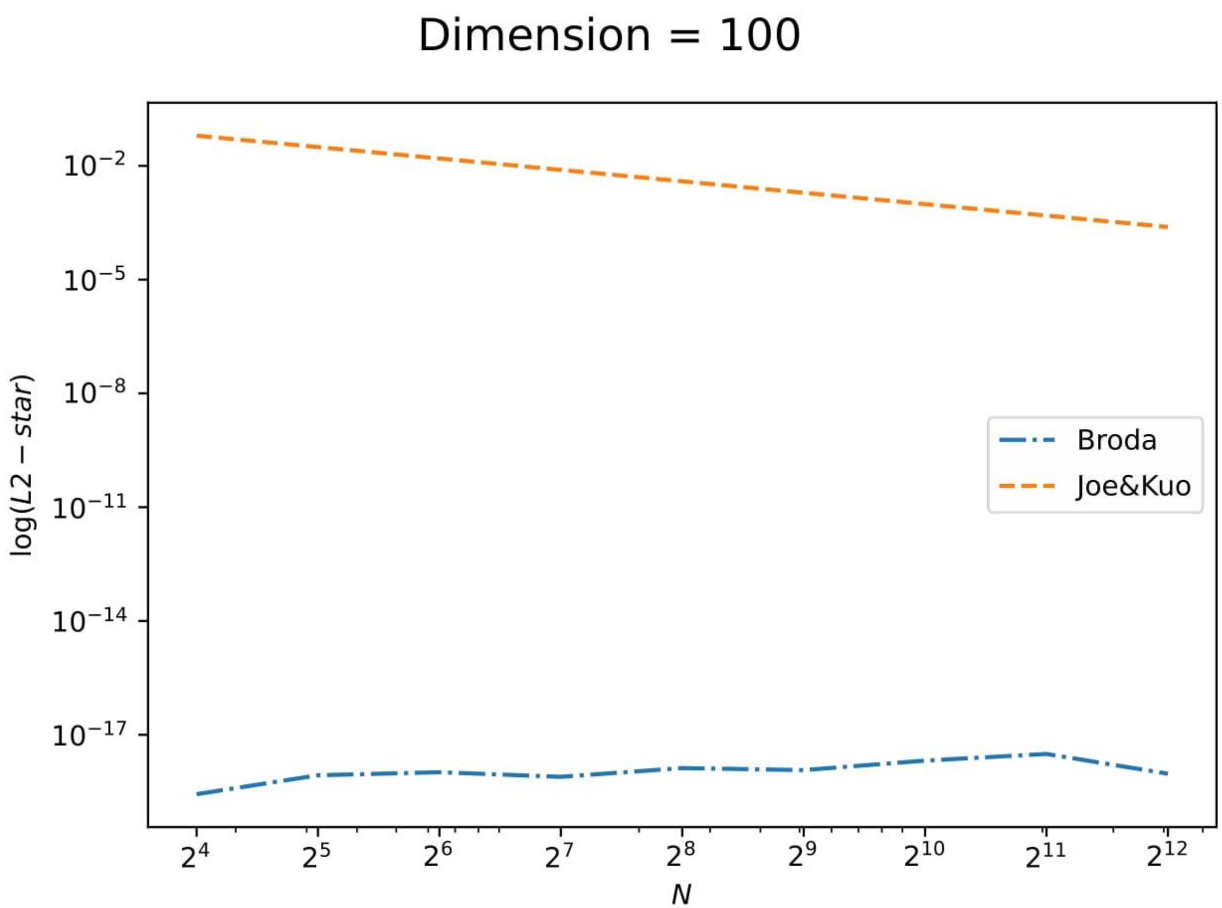
$$CD^2(\mathbf{X}_d^N) = \left(\frac{13}{12}\right)^d - \frac{2}{N}\sum_{i=1}^{N}\prod_{k=1}^{d}\left(1 + \frac{1}{2}|x_k^{(i)} - 0.5| - \frac{1}{2}|x_k^{(i)} - 0.5|^2\right)$$
$$+ \frac{1}{N^2}\sum_{i,j=1}^{N}\prod_{k=1}^{d}\left(1 + \frac{1}{2}|x_k^{(i)} - 0.5| + \frac{1}{2}|x_k^{(j)} - 0.5| - \frac{1}{2}|x_k^{(i)} - x_k^{(j)}|\right)$$



BRODA's generator – much lower Centered Discrepancy

L_2 discrepancy

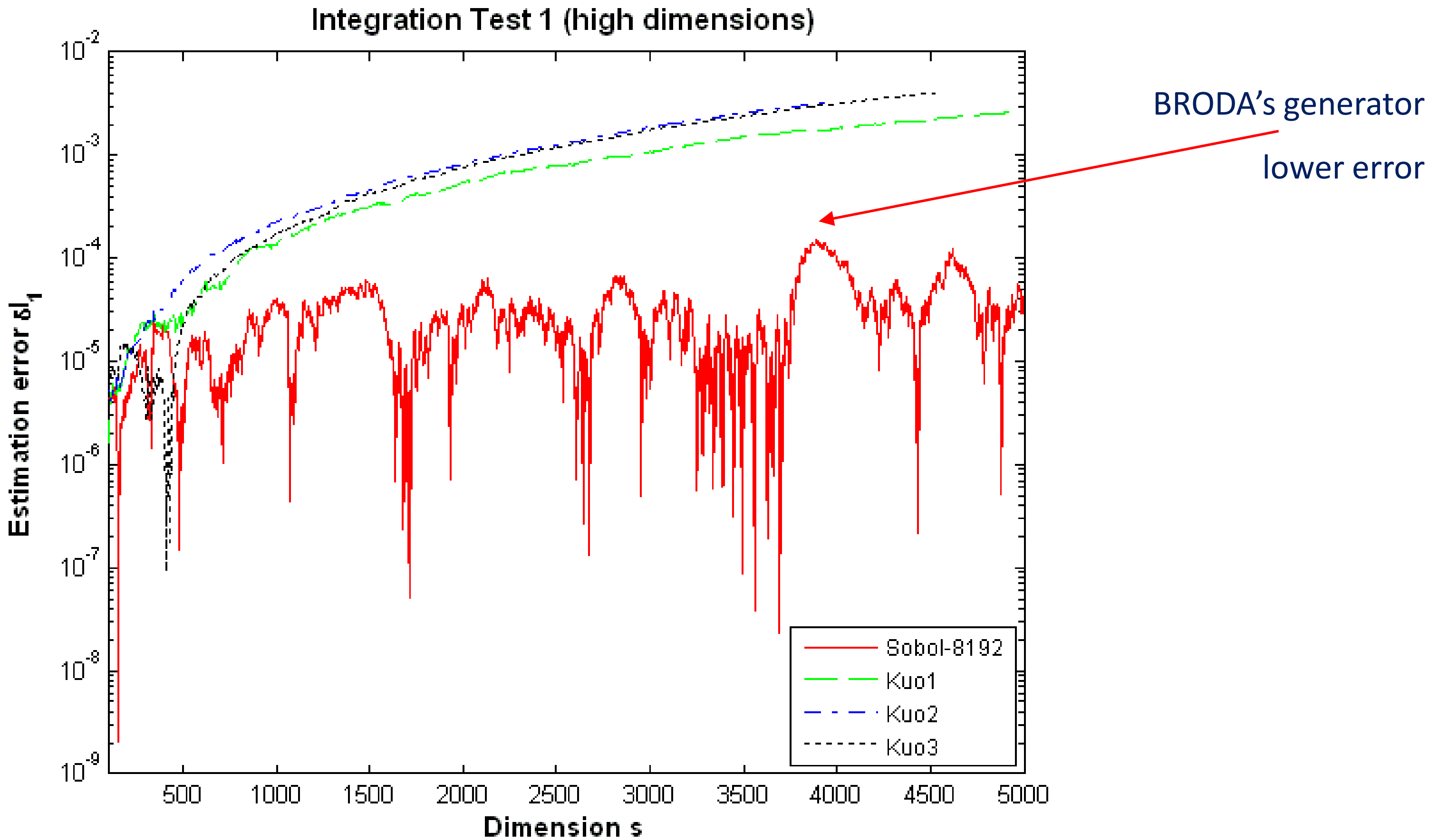
$$L_2\left(\mathbf{X}_d^N\right) = 3^{-d} - \frac{2^{1-d}}{N}\sum_{i=1}^{N}\prod_{k=1}^{d}\left(1 - x_l^{(k)^2}\right) + \frac{1}{N^2}\sum_{i,j=1}^{N}\prod_{k=1}^{d}\left(1 - \max\left(x_k^{(i)}, x_k^{(j)}\right)\right).$$



BRODA's generator – much lower L_2 Discrepancy

Integration test. I

$$I = \int_{[0,1]^n} \prod_{i=1}^{s} \left(1 + c_i\left(x_i - 0.5\right)\right) dx_i , \quad c_i = 0.01$$

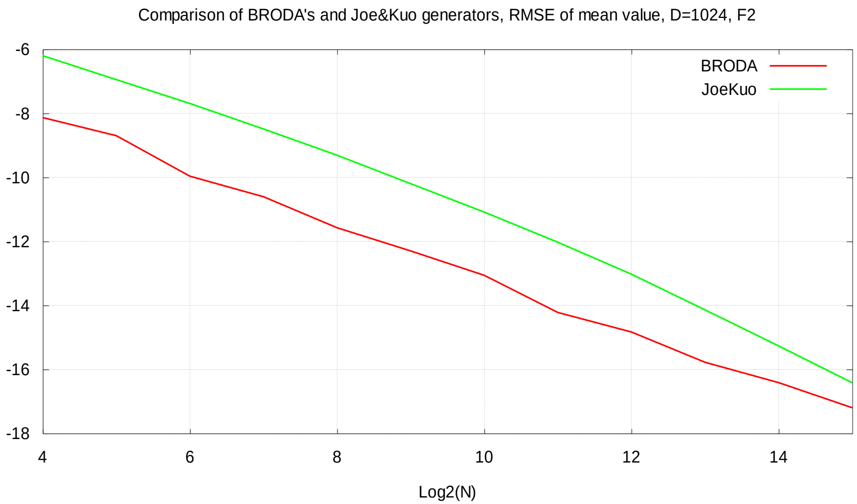


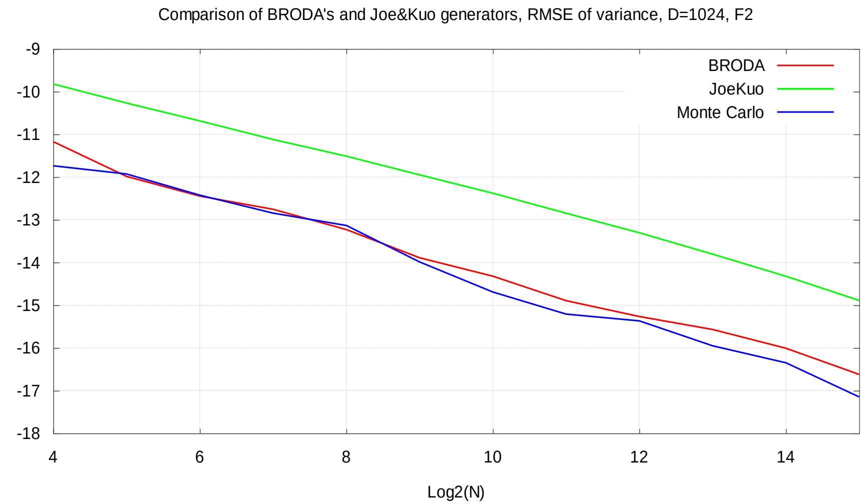
I. Sobol', D. Asotsky, A. Kreinin, S. Kucherenko. Construction and Comparison of High-Dimensional Sobol' Generators, 2011, Wilmott Journal, Nov, 64-79

Computation of function mean and variance. I

Mean value	Variance
$\mu_N = \frac{1}{N}\sum_{k=i}^{N} f(x_i)$	$V_N = \frac{1}{(N-1)}\sum_{k=i}^{N}(f(x_i)-\mu_N)^2$

Mean Value	Variance *V*
$(1+1/d)^d \int_{[0,1]^d} \prod_{i=1}^{d} x_i^{1/d} dx_i$	$\left[1+\frac{1}{d^2+2d}\right]^d - 1$





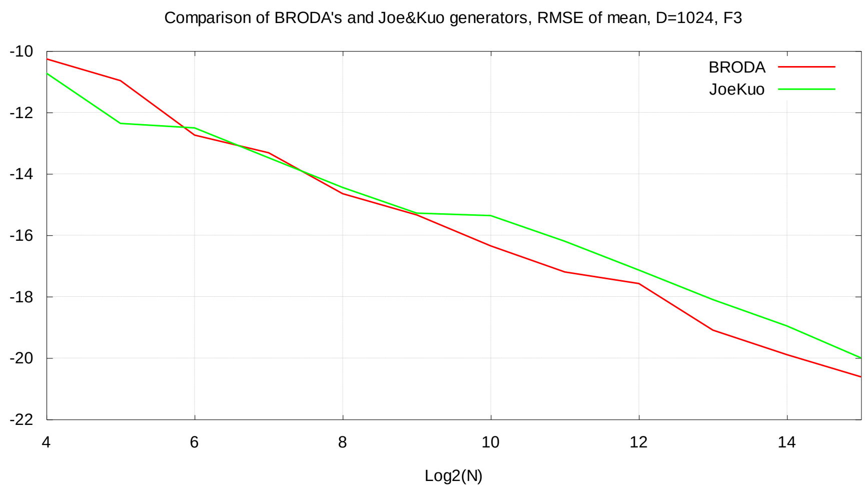
Computation of mean: low effective dimension.
Efficiency of BRODA is higher

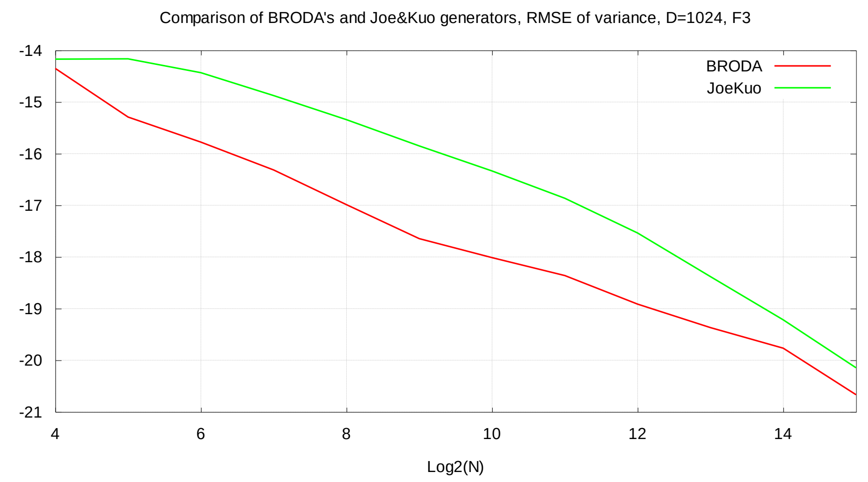
Computation of variance: high effective dimension
Efficiency of BRODA is higher

Computation of function mean and variance. II

Mean value	Variance
$\mu_N = \frac{1}{N}\sum_{k=i}^{N} f(x_i)$	$V_N = \frac{1}{(N-1)}\sum_{k=i}^{N}(f(x_i)-\mu_N)^2$

Mean Value	Variance *V*
$\frac{1}{(d-1/2)^d}\int_{[0,1]^d}\prod_{i=1}^{d}(d-x_i)dx_i$	$\left[1+\frac{1}{12(d-1/2)^2}\right]^d-1$





Computation of mean: low effective dimension.
Efficiencies of both generators are similar.

Computation of variance: high effective dimension
Efficiency of BRODA is higher

Spurious variance component

Consider $\bar{z}_d = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} z_i$, $\boldsymbol{z} = (z_1, ..., z_d) \sim N(0, I)$.

Such constructs are commonly used in pricing of path dependent options:

A terminal asset value $S(T)$ in the case of *d* time steps

$$S(T) = S_0 \, exp[\,(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{\Delta t}(z_1 + z_2 + \cdots + z_d)] =$$

$$= S_0 \, exp[\,(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\underline{\bar{z}_d}]$$

Wiener path is sampled using the Standard discretization scheme.

Variance $V(\bar{z}_d)$, assuming that $E(\bar{z}_d) = 0$:

$$V(\bar{z}_d) = \frac{1}{d}\sum_{i=1}^{d}\sum_{j=1}^{d}\rho_{ij} = \frac{1}{d}\left[\sum_{i=1}^{d} 1 + \sum_{i=1}^{d}\sum_{j:j\neq i}^{d}\rho_{ij}\right] = 1 + \bar{\rho}_d.$$

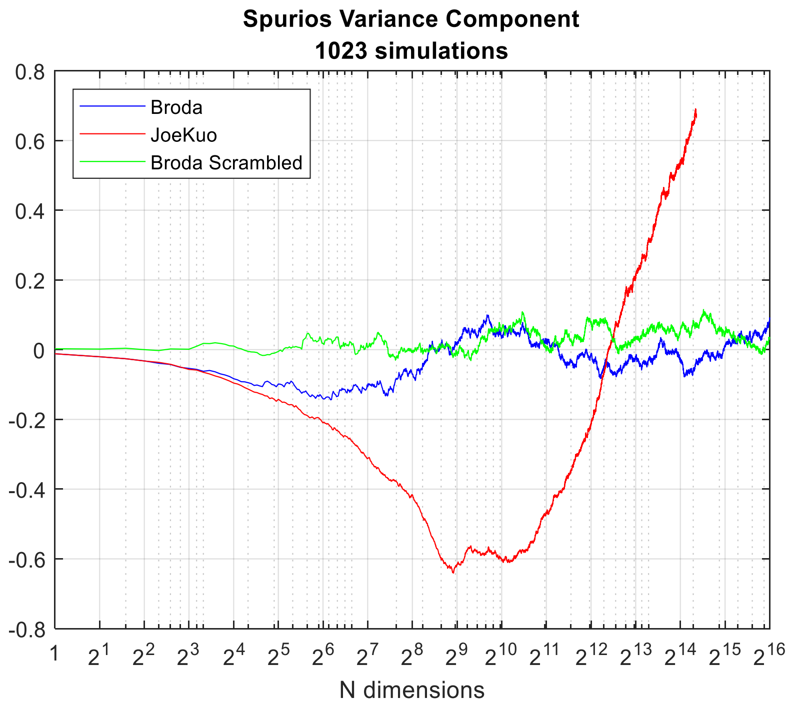
$\bar{\rho}_d = \frac{1}{d}\sum_{i=1}^{d}\sum_{j:j\neq i}^{d}\rho_{ij}$ is an average correlation, $\rho_{ij} = E[z_i z_j]$.

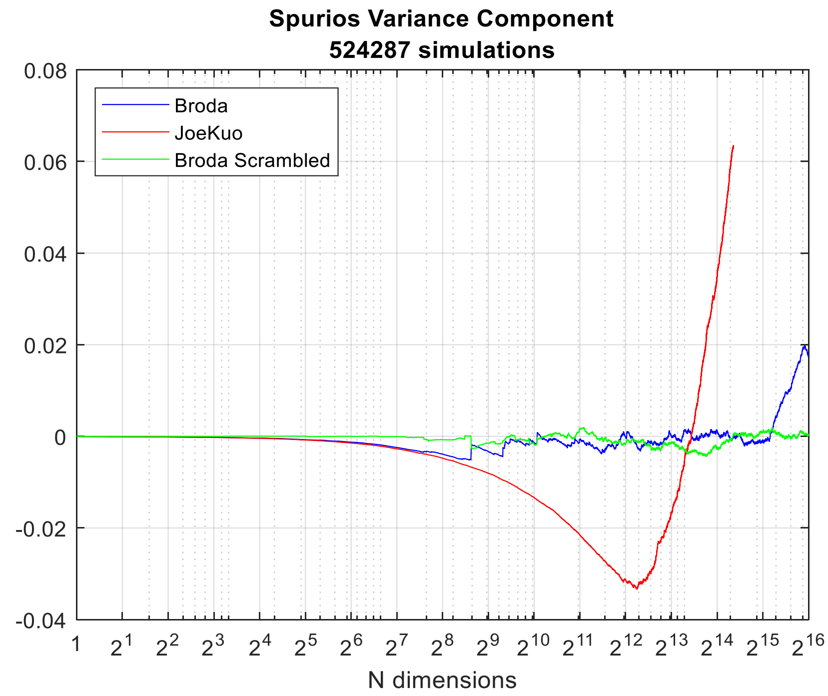
Theoretically $\rho_{ij} = 0, i \neq j$, hence $V(\bar{z}_d) = 1$.

Numerical $\bar{\rho}_d \neq 0$ due to the existence of spurious correlations between different dimensions; $z_i = N^{-1}(x_i)$, x_i's – are Sobol' points

We call $\bar{\rho}_d$ - a "spurious variance component"

Spurious variance component at different dimensions and number of points





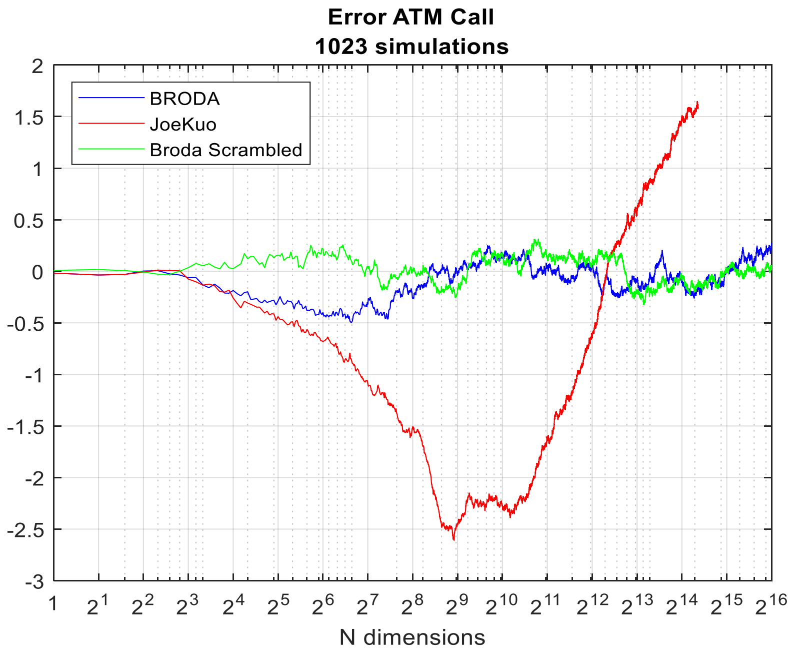
JoeKuo's generator has unacceptably high spurious variance component
BRODA's generator doesn't have this problem

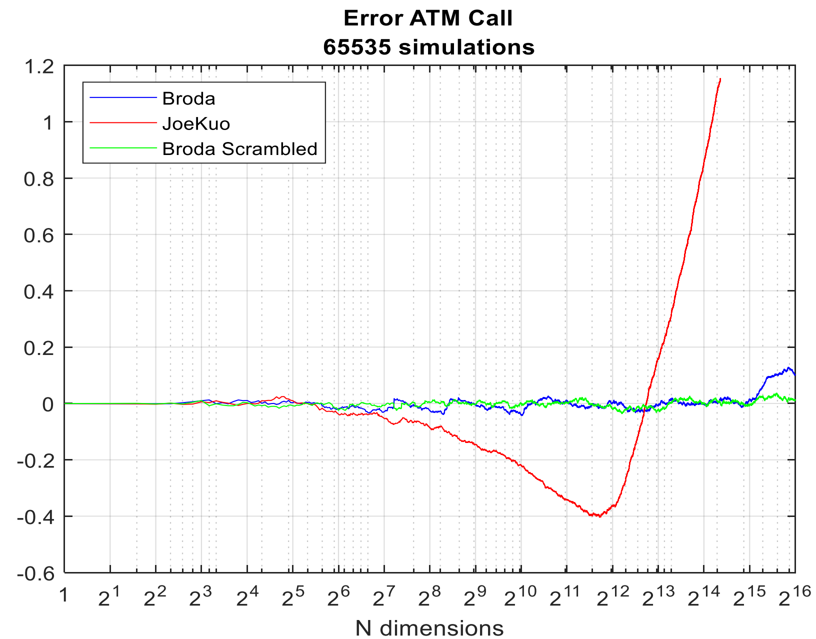
"Joe-Kuo's generator is good at minimizing *individual* Covariances, but is disastrously dismal in minimizing the impact of their *aggregate*. The Broda method(s) is optimized in some elaborate way to minimize the impact of *any* potential aggregate" (Fisglobal Ltd.)

Pricing of at the money call (ATM)

European call option: S_0 = 100, K= 100, r = 0.0, sigma = 0.2, T = 1y. C_theory(ATM) = 7.966.

$$\boldsymbol{C} = e^{-rT}\int_{H^d} max[0,(S(T)-K)]du_1\ldots du_d = e^{-rT}\int_{H^d} max[0,(S_0\,exp[(r-\tfrac{1}{2}\sigma^2)T+\sigma\sqrt{T}\bar{z}_d]-K)]du_1\ldots du_d = e^{-rT}\int_{H^d} max[0,(S_0\,exp[(r-\tfrac{\sigma^2}{2})t_i+\sigma\sqrt{\tfrac{T}{d}}\textstyle\sum_{j=1}^{i}\Phi^{-1}(u_j)]-K)]du_1\ldots du_d$$

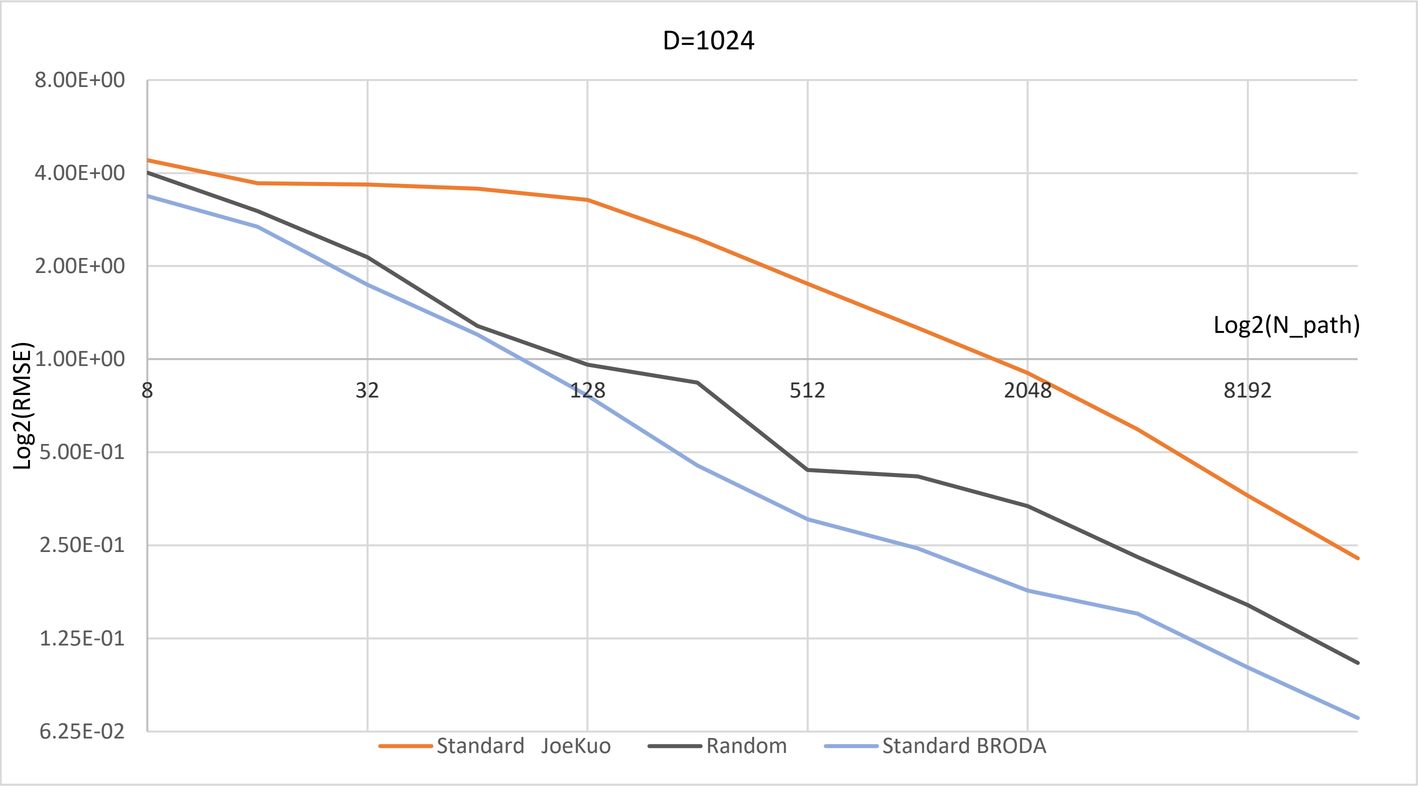




Pricing error versus the number of dimensions (time steps) at different number of sampled path

Joe&Kuo's generator exhibits high pricing error
BRODA's generator doesn't have this problem

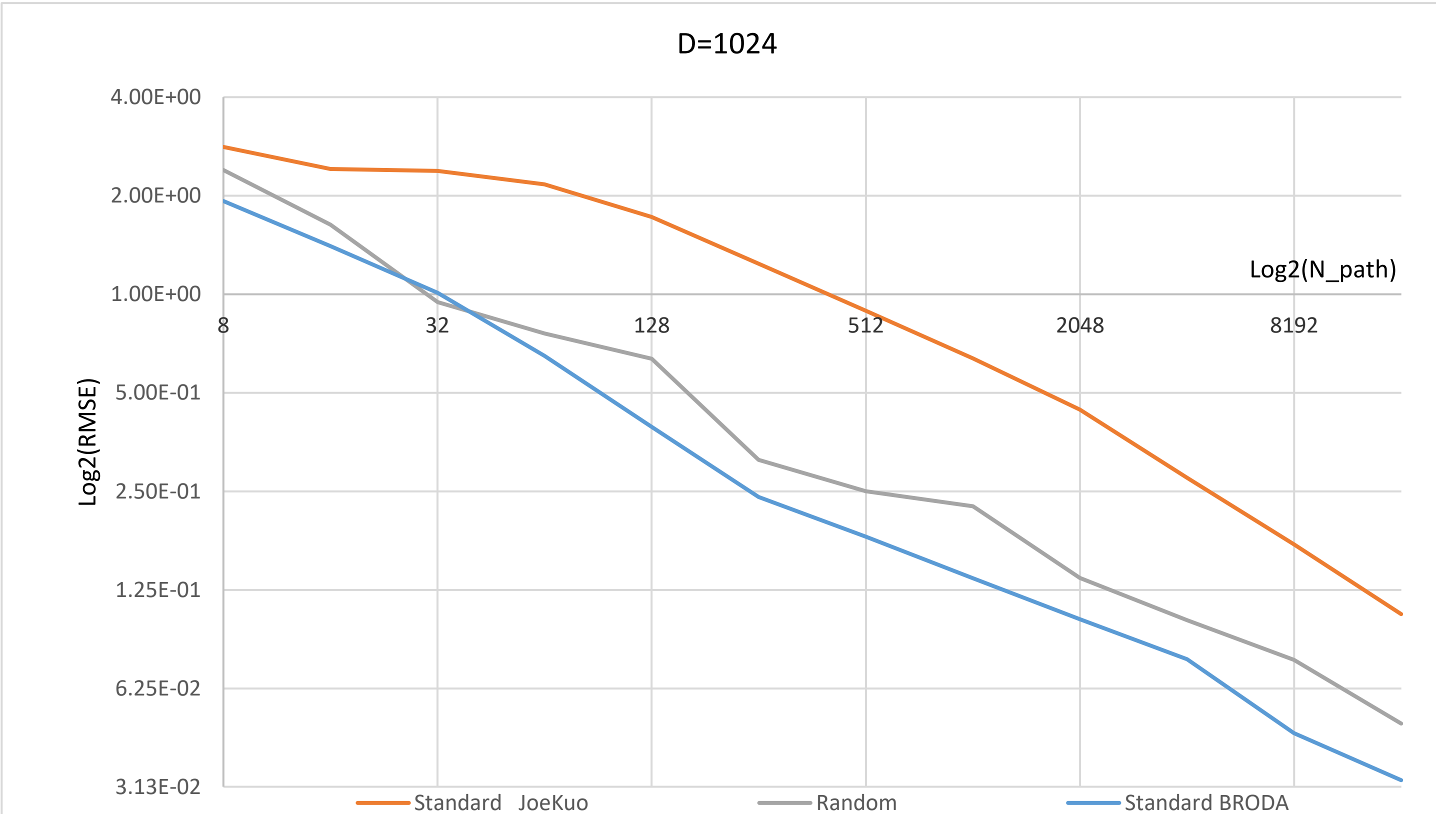
Pricing of at the money call (ATM). RMSE vrs N_path



Standard (incremental) sampling scheme. D = 1024 time steps.

Joe&Kuo's generator is much less efficient than BRODA's generator and MC

Pricing of the geometric Asian call. RMSE vrs N_path



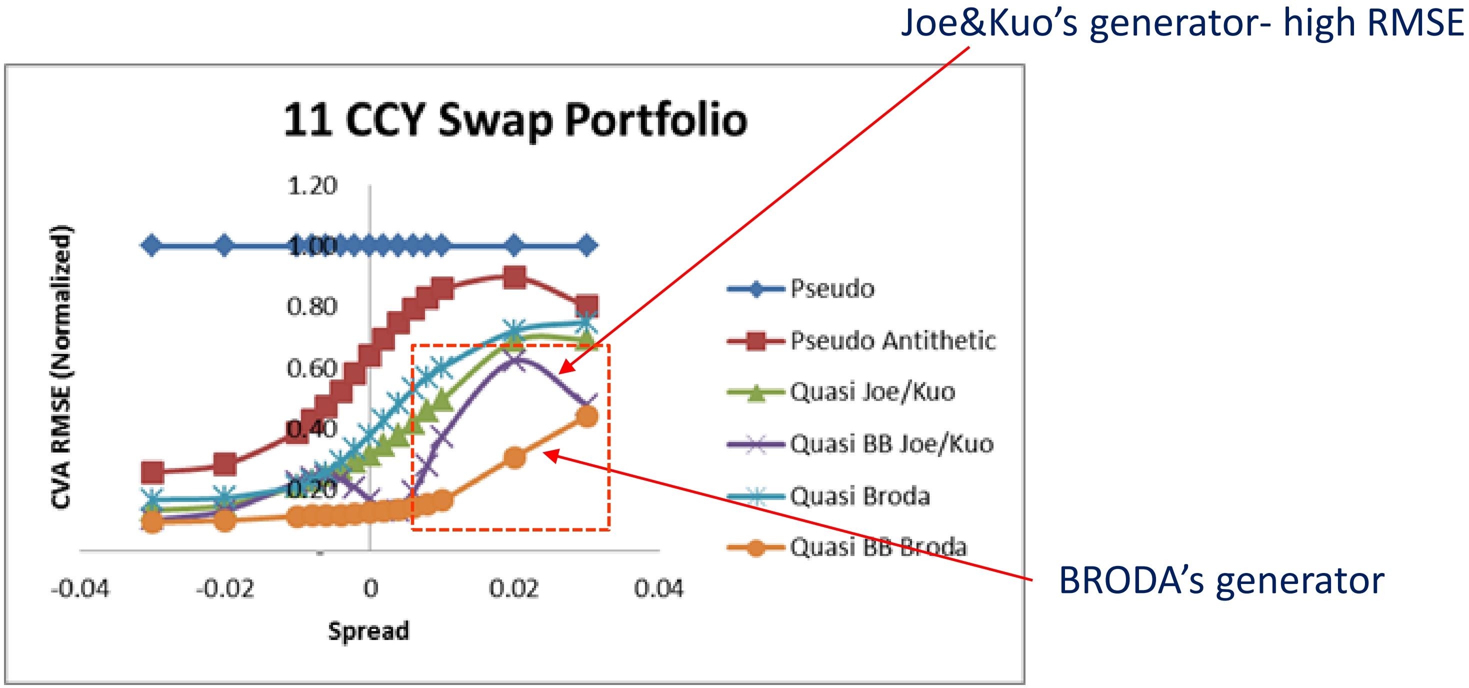
$S_0 = 100$, $K = 100$, $r = 0.05$, $\sigma = 0.2$, $T = 0.5$y, number of discrete time steps $d = 1024$

Standard (incremental) sampling:

$$C = e^{-rT} \int_{H^d} max[\,0, (\left[\prod_{i=1}^{d} S_0\, exp[\,(r - \frac{\sigma^2}{2})t_i + \sigma \sqrt{\frac{T}{d}} \sum_{j=1}^{i} \Phi^{-1}(u_j)]\right]^{1/d} - K)]du_1 \ldots du_d$$

Joe&Kuo's generator is much less efficient than BRODA's generator and MC

CVA and CVA Sensitivities*



CVA RMSE (normalized to pseudo error) on a 11 swap portfolio**, D=2541 dimensions.
Quasi BB .. = QMC+BB+PCA. Outer loop - time direction, inner loop - factor direction.
Quasi .. = QMC+STD+PCA. BB – Brownian Bridge

**all in different currencies as a function of the moneyness of the portfolio (fixed rate of the swaps, expressed as a spread over the par rate).

".. it appears that the Broda SobolSeq65536 direction numbers, when combined with the Brownian Bridge discretization, are more robust to portfolio changes, and we suspect, the effective dimension of the integrand."

*Stefano Renzitti, IHS Markit, Accelerating CVA and CVA Sensitivities Using Quasi-Monte Carlo Methods (2017). https://ssrn.com/abstract=3193219

Summary

	Joe&Kuo	BRODA
Efficiency	Low	High
Max Dimensionality	21201	131072*
Architecture	32 Bit	64 Bit
Additional uniformity properties	Optimised 2D projections	Property A for all dimensions and Property A' for adjacent dimensions
Support/ Updates	No	Yes

Comparison of Joe&Kuo and BRODA generators shows the superior performance of BRODA's Sobol' sequence generators

*both standard and scrambled versions

Acknowledgements: D. Asotsky, E. Atanassov, S. Renzitti, P. T. Roy

Publications related to applications of BRODA's SobolSeq generators

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3. M. Bianchetti, S. Kucherenko, S. Scoleri, Pricing and Risk Management with High-Dimensional Quasi Monte Carlo and Global Sensitivity Analysis, Wilmott, July, 46-70, 2015, http://www.broda.co.uk/doc/PricingRiskManagement_Sobol.pdf

4. M. Bianchetti, S. Kucherenko, S. Scoleri, High-Dimensional Quasi Monte Carlo and Global Sensitivity Analysis: Multi-Asset Options and AAD, WBS - 11th Fixed Income Conference, Paris, Oct 9, 2015 http://www.broda.co.uk/gsa/BKS-WBS-11thFixedIncomeConference-v5.pdf

5. S. Renzitti, IHS Markit, Accelerating CVA and CVA Sensitivities Using Quasi-Monte Carlo Methods (2017). https://ssrn.com/abstract=3193219

6. S. Renzitti, P. Bastani, S. Sivorot, S., 2020. Accelerating CVA and CVA Sensitivities Using Quasi-Monte Carlo Methods. Wilmott, 2020(108), pp.78-93. http://www.broda.co.uk/doc/IHS-MARKIT_Wilmott.pdf

7. D. Morgan, S. Kucherenko, Comparing Option Pricing Methods in q. Wilmott. 2020 Sep;2020(109): 58-69 http://www.broda.co.uk/doc/Morgan_Kucherenko_Wilmott.pdf https://doi.org/10.1002/wilm.10876

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9. S. Scoleri, M. Bianchetti S. Kucherenko. Application of Quasi Monte Carol and Global Sensitivity Analysis to Option Pricing and Greeks: Finite Differences vs. AAD. Wilmott. 2021 Nov;2021(116):66-83. https://www.broda.co.uk/doc/QMC_Basket_AAD.pdf

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