The importance of being scrambled: supercharged Quasi Monte Carlo

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Introduction and plan

Option pricing problem can be formulated as

$$I[f] = \mathbb{E}[f(x)] = \int_{H^d} f(x)dx, \quad (1)$$

where $\mathbb{E}[.]$ is the mathematical expectation and f the payoff function, integrable in the d-dimensional unit hypercube $H^d = [0,1]^d$.
The standard MC estimator is

$$I_N[f] = \frac{1}{N}\sum_{i=1}^{N} f(x_i), \quad (2)$$

where $\{x_i\}$ is a sequence of random points in H^d of length N.

By law of large number, a.s, we have $I_N[f] \longrightarrow I[f]$ and central limit theorem provides confidence interval:

$$I[f] \in [I_N[f] \pm c\frac{\hat{\sigma}_N}{\sqrt{N}}] \quad (3)$$

with probability close to $1-\alpha_c$ for N large enough

with $\hat{\sigma}_N := \sqrt{\frac{1}{N-1}\sum_{n=1}^{N}(f(x_i) - I_N[f])^2}$ and $\alpha_c = \mathbb{P}[|X| > c]$ for $X \sim N(0,1)$.

Introduction and plan

MC convergence rate does not depend on the number of variable d but it is rather slow.

$\rightarrow$ In many financial applications, Quasi Monte Carlo (QMC) based on Sobol low-discrepancy sequences (LDS) outperforms Monte Carlo showing faster and more stable convergence:

- Instead of random, LDS are specifically designed to place points $\{x_i\}$ deterministically as uniformly as possible. Successive LDS points know about the position of previously sampled points and fill the gaps between them;
- $I_N[f] \longrightarrow I[f]$ as rate $O(N^{-\alpha})$ with $0.5 < \alpha \leq 1$ depending on the effective dimensionality of the underlying problem.

However, unlike MC, QMC lacks a practical error estimate.

$\rightarrow$ Randomized QMC (RQMC) method combines the best of two methods: it randomizes the points $\{x_i\}$ by preserving their low discrepancy property and allows to compute confidence intervals around the estimated value, providing a practical error bound.

Introduction and plan

Plan:

- RQMC and SobolSeq generators;
- Hyperbolic local volatility model, discretization schemes, MC pricing and Greek;
- Numerical results: pricing and risk comparison between MC, Sobol QMC, and RQMC (Owen scrambling and digital shift);
- Summary.

QMC Estimator

QMC estimator:

$$I_N[f] = \frac{1}{N}\sum_{i=1}^{N} f(Q_i), \tag{4}$$

LDS points $\{Q_i\}$, $Q_i \in H^d$

Pros: High rate of (asymptotic) convergence: $O(N^{-1})$ versus $O(N^{-0.5})$ for MC

Cons: No practical estimates of the integration error: the Koksma-Hlawka inequality $\epsilon_{QMC} = |I_N[f] - I[f]| \leq V(f)D_N$ is too conservative and not practical.

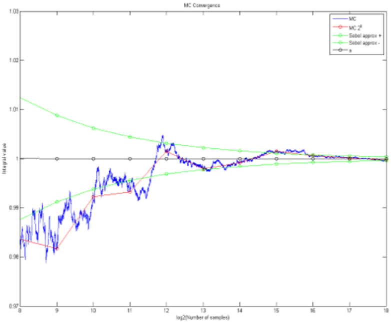
We want to have practical confidence intervals similar to MC

Example: MC with confidence intervals

$$I[f] = (1 + 1/d)^d \int_{[0,1]^d} \prod_{i=1}^{d} x_i^{1/d} dx_i = 1.0 \quad (5)$$

$$\mathbb{P}\left[I_N[f] - 0.6745 \frac{\sigma_{MC}}{\sqrt{N}} < I(f) < I_N[f] + 0.6745 \frac{\sigma_{MC}}{\sqrt{N}} \right] = 0.5 \quad (6)$$

$$\sigma_{MC} = \left[\frac{1}{(N-1)} \sum_{i=1}^{N} (f(X_i) - I_N(f))^2 \right]^{1/2}. \quad (7)$$



Randomised QMC Estimator

A set of K randomised replication of $\{Q_i\}$: $\{V_i\} = V_i^k, k = 1, ..., K$.
$\hat{\mu}_n^k$ - the $k - th$ RQMC estimator for (1):

$$\hat{\mu}_n^k = \frac{1}{n} \sum_{i=1}^{n} f(V_i^k), \tag{8}$$

$\hat{\mu}_n^k$ are i.i.d. random variables

The RQMC sample mean

$$\bar{\mu}_n = \frac{1}{K} \sum_{k=1}^{K} \hat{\mu}_n^k. \tag{9}$$

TABLE: MC and RQMC sample standard deviations σ, RMSE errors ε and confidence intervals. The total number of function evaluations $N = nK$.

$\sigma_{MC} = \sqrt{\frac{1}{(N-1)} \sum_{i=1}^{N} (f(X_i) - I_N[f])^2}$	$\sigma_{RQMC} = \sqrt{\frac{1}{(K-1)} \sum_{k=1}^{K} (\hat{\mu}_n^k - \bar{\mu}_n)^2}$
$\varepsilon_{MC} = \frac{\sigma_{MC}}{\sqrt{N}}$	$\varepsilon_{RQMC} = \frac{\sigma_{RQMC}}{\sqrt{K}}$
$I_N[f] \pm z_{\delta/2} \varepsilon_{MC}$	$\bar{\mu}_n \pm z_{\delta/2} \varepsilon_{RQMC}$

K is large enough - $\bar{\mu}_n \sim N(I[f], \sigma_{RQMC})$.

z_δ is δ quantile: $F(z_\delta) = 1 - \delta$.

Example: For a 95% confidence interval, $\delta = .05$, $z_{\delta/2} \approx 1.96$.

Randomization with Owen's scrambling

Base b-expansion of an LDS point $Q_i \in H^d$:

$$Q_i^j = \sum_{p=1}^{m} q_{i,p}^j b^{-p}, \tag{10}$$

$j = 1, ..., d,\ i = 1, ..., N,\ N = b^m$,
$b \geq 2,\ q_{i,p}^j \in \{0, 1 \ldots, b-1\}$.

Coefficients $q_{i,p}^j$ are permuted:

$v_{i,1}^j = \pi^j(q_{i,1}^j)$, $v_{i,2}^j = \pi^j_{(q_{i,1}^j)}(q_{i,2}^j)$, $v_{i,3}^j = \pi^j_{(q_{i,1}^j, q_{i,2}^j)}(q_{i,3}^j)$, ...

Owens' scrambled version $V_i^j = \sum_{p=1}^m v_{i,p}^j b^{-p}$ of Q_i^j

Uniform random permutations π^j over the set of $\{0, 1 \ldots, b-1\}$ are

1) mutually independent

2) each of them depends on previous leading digits of Q_i^j (Owen 1997)[1]

[1] A.Owen. The Annals of Statistics, 25(4):1541, 1997.

Randomization with Owen's scrambling. Pros and Cons

Pros: For sufficiently smooth functions $\varepsilon_{RQMC} \sim O(1/(n^{(3/2-\alpha)}))$

$\sqrt{n}$ times higher than QMC rate $O(1/(n^{(1-\alpha)}))$.

This reduction arises from random error cancellations.

Cons: Permutation tree $\Pi \sim d(b^M - 1)/(b-1)$ permutations.

Example: Sobol' LDS: $b = 2$, $M = 32$, dimension $d = 100$ - Permutation tree $\Pi = (\pi, \pi_0, \pi_1, \pi_{00}, \pi_{01}, \pi_{10}, \pi_{11}, ..), \forall j$ - need to store in memory $\sim 4.3\ 10^{11}$ permutations.

BRODA developed a modification of Owen's scrambling with additional permutations[2]. It has reduced memory and CPU requirements.

[2] E.Atanassov, S. Kucherenko. Implementation of Owen's scrambling with additional permutations for Sobol' sequences. BRODA Ltd., UK, 2021

Randomization with Digital Shift

A set d-dimensional Sobol' points $\{Q_i\}$ in base $b = 2$

$$Q_i^j = \sum_{p=1}^{m} q_{i,p}^j 2^{-p}, \tag{11}$$

Generate r.n. $U \sim U[0,1]^d$, $U^j = \sum_{p=1}^{m} u_p^j 2^{-p}$

Randomised version $V_i^j = \sum_{p=1}^{m} v_{i,p}^j 2^{-p}$:

$$v_{i,p}^j = (q_{i,p}^j \oplus u_p^j)$$

$\oplus$ - binary addition modulo 2 (a bitwise XOR operator):
$0 \oplus 0 = 0$; $1 \oplus 1 = 0$; $0 \oplus 1 = 1$; $1 \oplus 0 = 1$

Randomization with Digital Shift. Pros and Cons

K randomised replicas of $\{Q_i\} \rightarrow$ same set of $\{Q_i\}$ with different U^k.

Pros: A. Satisfies statistical r.v. and LDS properties of RQMC.
B. Simple to implement and does not impose extra memory requirements.

Cons: Does not possess the increased rate of Owen's scrambling ($\varepsilon_{RQMC} \sim O(1/(n^{(3/2-\alpha)}))$).

EXAMPLE. DIGITAL SHIFT

A set 2-dimensional Sobol' LDS points in base $b = 2, M = 6$:

$$Q_i^j = \sum_{p=1}^{6} q_{i,p}^j 2^{-p}, \tag{12}$$

Coefficients of binary expansion $\{q_{i,p}\} \in \{0, 1\}$:

	x-coord.	y-coord.
$(\frac{1}{64}, \frac{51}{64})$	000001	110011
$(\frac{33}{64}, \frac{39}{64})$	100001	010011
$(\frac{17}{64}, \frac{3}{64})$	010001	000011
$(\frac{49}{64}, \frac{35}{64})$	110001	100011
$(\frac{9}{64}, \frac{27}{64})$	001001	011011
$(\frac{41}{64}, \frac{59}{64})$	101001	111011
$(\frac{25}{64}, \frac{43}{64})$	011001	101011
$(\frac{57}{64}, \frac{11}{64})$	111001	001011

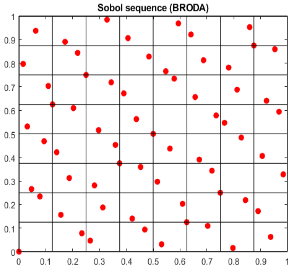
Digital Shift. Example

Generate r.n. U. Assume $\{u_p^j\}$ in base $b = 2$: $U_x = 100101$, $U_y = 000110$
Recall $0 \oplus 0 = 0$; $1 \oplus 1 = 0$; $0 \oplus 1 = 1$; $1 \oplus 0 = 1$;

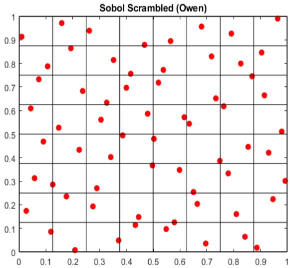
Applying $\oplus$	x_1-comp.	y_1-comp.
	000001	110011
	100101	000110
	100100	110101

	x-coord.	y-coord.			x-coord.	y-coord.
$(\frac{1}{64}, \frac{51}{64})$	000001	110011		$(\frac{36}{64}, \frac{53}{64})$	100100	110101
$(\frac{33}{64}, \frac{39}{64})$	100001	010011		$(\frac{4}{64}, \frac{21}{64})$	000100	010101
$(\frac{17}{64}, \frac{3}{64})$	010001	000011		$(\frac{52}{64}, \frac{5}{64})$	110100	000101
$(\frac{49}{64}, \frac{35}{64})$	110001	100011	$\longrightarrow$	$(\frac{20}{64}, \frac{37}{64})$	010100	100101
$(\frac{9}{64}, \frac{27}{64})$	001001	011011		$(\frac{44}{64}, \frac{29}{64})$	101100	011101
$(\frac{41}{64}, \frac{59}{64})$	101001	111011		$(\frac{12}{64}, \frac{61}{64})$	001100	111101
$(\frac{25}{64}, \frac{43}{64})$	011001	101011		$(\frac{60}{64}, \frac{45}{64})$	111100	101101
$(\frac{57}{64}, \frac{11}{64})$	111001	001011		$(\frac{28}{64}, \frac{13}{64})$	011100	001101

Comparison of Standard and Scrambled Sobol Sequences



(a) SobolSeq



(b) SobolOwen

Comparison of Random numbers and Sobol Sequences - Youtube, BRODA
Difference between Standard and Scrambled Sobol Sequences - Youtube, BRODA

The efficiency of Sobol' LDS generator depends on direction numbers. Badly initialized direction numbers → poor 2D projections at low number of N[3].

Joe&Kuo's generator - 'optimized' 2D projections (maximum dim. d =21201)[4].

BRODA's SobolSeq - additional uniformity properties[5]:
1) Property A for all dimensions (maximum dim. d =131072)
2) Property A' for adjacent dimensions.

BRODA's SobolSeq generator outperforms Joe&Kuo's generator

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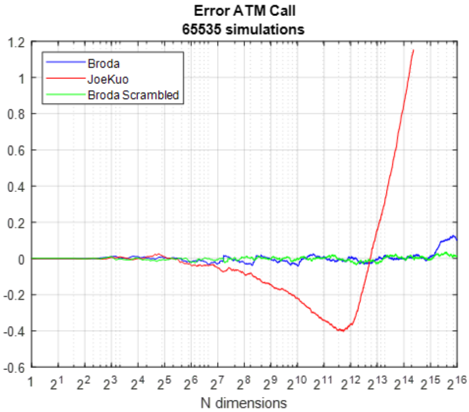
[3] P. Jackel, Monte Carlo Methods in Finance, John Wiley&Sons, 2002
[4] S.Joe, F.Y.Kuo. SIAM J. Scientific Comp., 30, 2635-2654, 2008
[5] I. Sobol', D. Asotsky, A. Kreinin, S. Kucherenko. 2011, Wilmott Journal, Nov, 64-79

Comparison of JoeKuo and BRODA SobolSeq

European call: $S_0 = 100$, K= 100, r $= 0.0$, $\sigma = 0.2$, T $=$ 1y. $C_{theory} = 7.966$.

$$C = e^{-rT}\int_{H^d} \max[0, (S_0 \exp[(r - \frac{\sigma^2}{2})T + \sigma\sqrt{\frac{T}{d}}\sum_{j=1}^{d}\Phi^{-1}(u_j)] - K)]du_1...du_d$$



BRODA - SobolSeq

BRODA Scrambled - Owen scrambling with additional permutations

JoeKuo - Direction numbers of Joe&Kuo

Time homogeneous HLV model

Significant sensitivity of exotic price to market volatility skew (see e.g Gatheral 06)

$\Longrightarrow$ Skew models: time-homogeneous hyperbolic local volatility model (Jackel 2006)

$$dS_t = rS_t dt + \tilde{\sigma}(S_t) dW_t,\ \ S_0 = 1, \tag{13}$$

with the risk free interest rate and

$$\tilde{\sigma}(S) = \nu \Big\{ \frac{(1-\beta+\beta^2)}{\beta} S_t + \frac{(\beta-1)}{\beta} \Big(\sqrt{S_t^2 + \beta^2(1-S_t)^2} - \beta \Big) \Big\} \tag{14}$$

and $\nu > 0$ the level of volatility and $\beta \in (0, 1]$ the skew parameter.

Time homogeneous HLV model

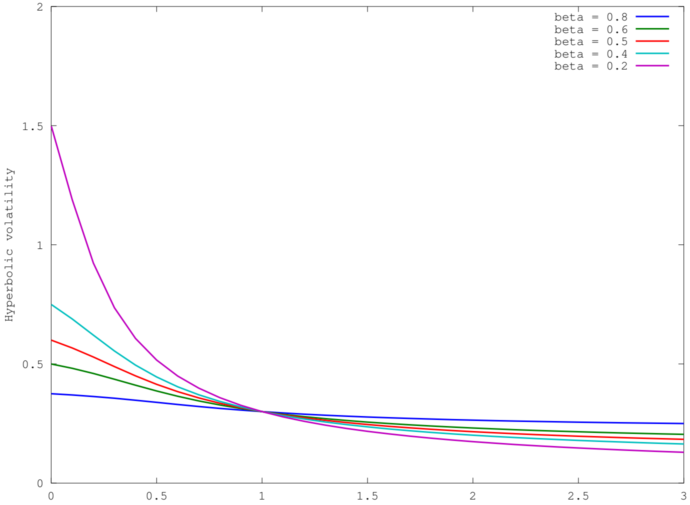


Figure: Impact of the value β on the hyperbolic local volatility for fixed volatility level $\nu = 0.3$.

Euler time discretization of the SDE

$$dY(t) = [r - \frac{1}{2}\sigma^2(Y(t))]dt + \sigma(Y(t))dW_t, \ \ Y(0) = \log(S(0)), \tag{15}$$

with $Y(t) = \ln(S(t))$ and $\sigma(Y) = \frac{\tilde{\sigma}(e^Y)}{e^Y}$.

$$Y^n(t_{i+1}) = Y^n(t_i) + [r - \frac{1}{2}\sigma^2(Y^n(t_i))](t_{i+1} - t_i) + \sigma(Y^n(t_i))(W(t_{i+1}) - W(t_i)) \tag{16}$$

with $Y^n(0) = \log(S(0))$, $\Delta t = \frac{T}{n}, t_i = i\Delta t,\ i = 0, .., n$.

Discretization of the Wiener process

- Standard (incremental) discretization:

$$W(t_i) = W(t_{i-1}) + \sqrt{\Delta t} Z_i \ \ 1 \leq i \leq n, \tag{17}$$

(Z_i) independent standard normal variates.

Remark:

$$Z_i \stackrel{L}{=} N^{-1}(U_i), \quad U_i \rightarrow \mathbb{U}[0,1] \tag{18}$$

- Brownian bridge algorithm: it is based on the use of conditional distributions
  1. First we generate the variable at the terminal point
  $$W(T) = \sqrt{T} Z_1 \tag{19}$$
  2. Then we fill other points using already found values of $W(t_i)$
  $$W(t_i) = (1-\gamma)\, W(t_l) + \gamma\, W(t_m) + \sqrt{\gamma(1-\gamma)(m-l)\Delta t} Z_i, \tag{20}$$
  where $\gamma = \frac{i-l}{m-l}$ with $l \leq i \leq m$.
  $$V(\, W(t_i) |\, W(t_l),\, W(t_m)) = \gamma(1-\gamma)(m-l)\Delta t. \tag{21}$$
  It decreases at the successive levels of refinement and the first few points contain the most of the variance.

Both algorithms generate discrete Brownian motion path for Euler scheme. However, QMC and RQMC algorithms have different efficiencies with the Brownian bridge algorithm getting a much higher convergence rate.

Monte Carlo simulation of Asian option and computation of Greeks

Geometric average Asian call option with payoff

$$P_A = \max(\bar{S} - K, 0), \tag{22}$$

with $\bar{S} = (\prod_{i=1}^{n} S_i)^{\frac{1}{n}}$ where S_i is the asset price at time $t_i = i\frac{T}{n}$, $1 \leq i \leq n$.

$\Longrightarrow$ High dimensionality problem.

Pricing and sensitivity factor Δ are given by

$$AC(T,K) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[P_A] \approx AC_N(T,K) = e^{-rT}\left[\frac{1}{N}\sum_{i=1}^{N}\max(\bar{S}^{(i)} - K, 0)\right]. \tag{23}$$

$$\Delta = \frac{\partial AC(T,K)}{\partial S(0)} \approx \frac{AC_N(T,K,S(0)+\epsilon_s) - AC_N(T,K,S(0)-\epsilon_s)}{2\epsilon_s} \tag{24}$$

where $\bar{S}^{(i)}$ is an approximation of $\bar{S}$ using the simulated price paths i.

Numerical results:price and delta convergence

Parameters: $S_0 = 100$, $r = 3\%$, $T = 1$, $\nu = 30\%$, $\beta = 0.5$, number of discrete time steps $d = 256 \implies$ high dimension problem.

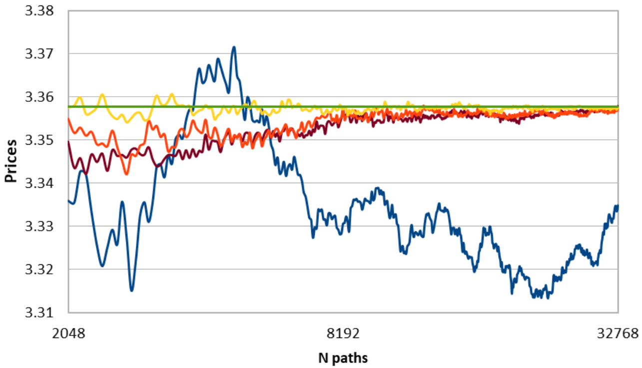


Figure: Asian ATM call price w.r.t number of paths N

Numerical results:price and delta convergence

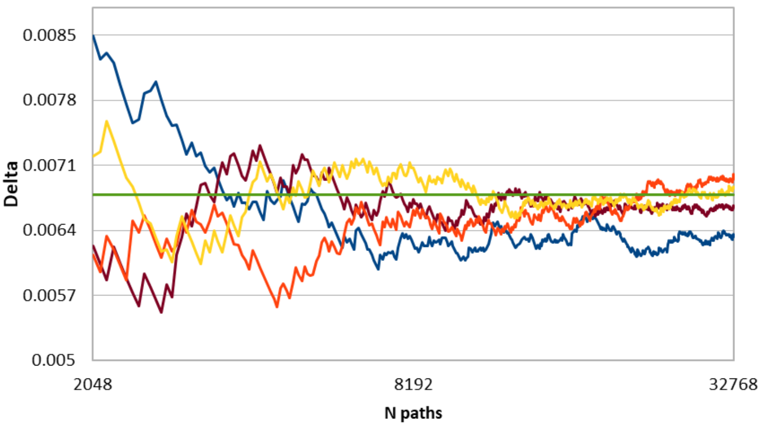


Figure: Asian OTM call delta w.r.t number of paths N

Numerical results: confidence intervals for prices

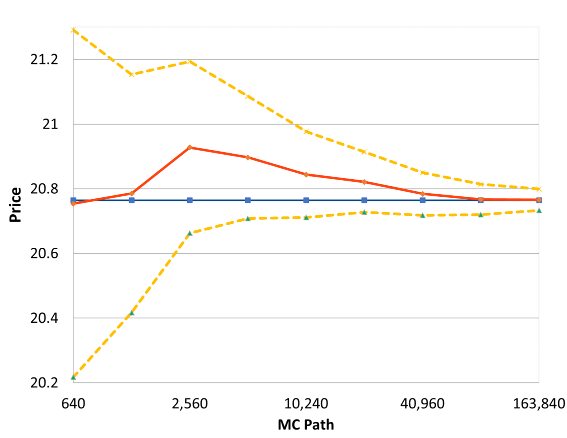


Figure: Prices and confidence intervals for ITM call for MC method versus the number of MC paths (in $log2$ scale)

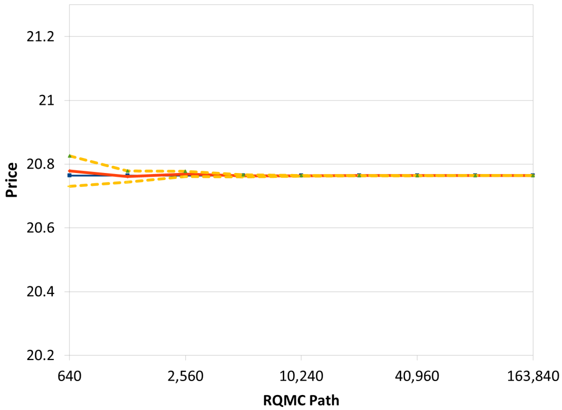


Figure: Prices and confidence intervals for ITM call for RQMC method versus the number of RQMC paths (in $log2$ scale) at $K = 10$

Numerical results: Confidence intervals for prices

TABLE: ε_{MC} and ε_{RQMC} at K=10, $n=2^{14}$ (N=163840). The ratio of MC to RQMC error estimates is given for RQMC with Owen's scrambling

	ITM	ATM	OTM
ε_{MC}	$1.7\ 10^{-2}$	$1.1\ 10^{-2}$	$1.4\ 10^{-2}$
ε_{RQMC} (Owen)	$7.0\ 10^{-5}$	$2.7\ 10^{-4}$	$1.2\ 10^{-4}$
ε_{RQMC} (DS)	$1.5\ 10^{-4}$	$1.8\ 10^{-4}$	$8.4\ 10^{-5}$
Ratio	243	40	116

Power law integration error approximation:

$$\varepsilon \sim \frac{C}{N^{\alpha}} \ . \tag{25}$$

Use the RMSE below to approximate the rate of convergence

$$\varepsilon_N = \sqrt{\frac{1}{K}\sum_{k=1}^{K}\left(V - V_N^{(k)}\right)^2}, \tag{26}$$

Numerical results: performance analysis

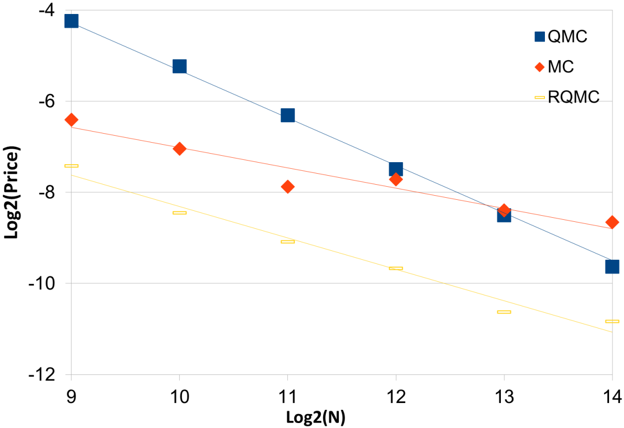


Figure: RMSE for ITM Asian call Prices w.r.t number of paths

Numerical results: performance analysis

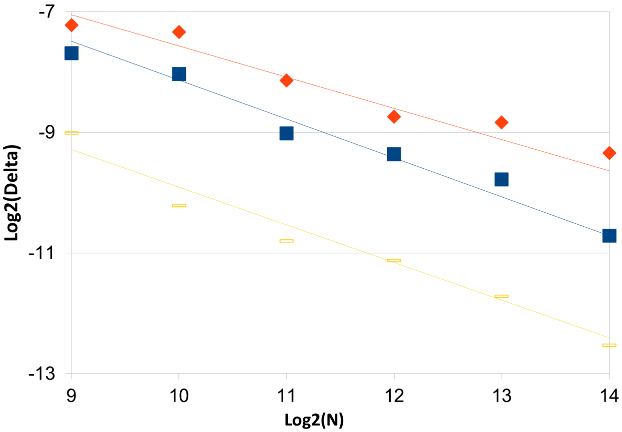


Figure: RMSE for ATM Asian call Deltas w.r.t number of paths

- Present and discuss the results of application of MC, QMC and RQMC methods for derivative pricing and risk analysis using the hyperbolic local volatility model;
- Results on Asian option show the superior performance of the QMC and RQMC methods;
- Application of effective dimension reduction scheme s.t Brownian bridge or PCA is critical and improves dramatically the efficiency of QMC and RQMC methods based on Sobol' sequences in comparison with the standard (incremental) construction;
- RQMC not only increases the rate of convergence of QMC but also allow to compute confidence intervals around the estimated value. Efficiency of RQMC strongly depend on the scrambling methods. We recommend using Sobol' LDS with Owen's scrambling.